ON THE EXISTENCE OF EQUILIBRIUM IN BAYESIAN GAMES WITHOUT COMPLEMENTARITIES

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Abstract. This paper presents new results on the existence of Bayesian equilibria in pure strategies of a specified functional form. These results broaden the scope of methods developed by Reny (2011) well beyond monotone pure strategies. Applications include natural models of first-price and all-pay auctions not covered by previous existence results. To illustrate the scope of our results, we present two applications to auctions: (i) a first-price auction in which bidders' payoffs have a very general interdependence structure; and (ii) an all-pay auction with non-monotone equilibrium.

Keywords: Bayesian games, monotone strategies, pure-strategy equilibrium, auctions.

1. Introduction

This paper analyzes a class of games of incomplete information in which each player has private information about her *type*, chooses an *action*, and receives a *payoff* as function of the profiles of types and actions. There is a *common prior* on the space of type profiles, so each agent's beliefs after observing her type are derived by Bayesian conditioning. Players may be heterogeneous in their preferences and type distributions. Moreover, players' types may not be independent and their payoff

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functions may depend directly on all players' types. Auctions are perhaps the most important games in such class, but our formulation is very general and most other models of interest have this structure.

The goal of this paper is to generalize some of the assumptions required in the existing literature on existence of pure strategy Nash equilibria. In particular, previous existence results require some version of the following assumptions:

(1) "weak quasi-supermodularity:" informally, the coordinates of a a player's own action vector need to be complementary; and
(2) "weak single-crossing:" informally, a player's incremental returns of actions are nondecreasing in her types.

We show that pure-strategy equilibria exist under significantly more general conditions, without impeding the analyst's ability to describe the properties of the equilibrium. The class of Bayesian games we cover includes games in which the players' action vectors are substitutes, and players' incremental returns of actions are not always increasing in their types. Despite the generality of these games, pure-strategy equilibria are well-behaved, in strategies that belong to a particular class of interest, such as the set of functions of bounded total variation, or functions of mixed monotonicity.[1]

The approach we adopt in this paper is motivated by positive questions. The goal is to develop a model that introduces new considerations to the analysis of Bayesian games and provides useful (testable) predictions. In the context of auctions in particular, we seek a convenient modeling tool for describing bidders' behavior in environments where weak quasi-supermodularity and weak single-crossing are too strong or unlikely to be true. Providing a more comprehensive theoretical framework for interpreting data has important implications for empirical and experimental research on auctions. In experimental work, it is usually the case that the questions of interest cannot be answered empirically until an internally consistent model of an auction game is specified. Thus our result extends the kind of economic questions that can be investigated using traditional experimental methods.[2] Further, structural econometric approaches to auctions have been mostly restricted to a limited class of models, usually settings with one object in which the equilibrium bidding strategies are monotone. There are few extensions to environments with multiple objects, with

[1] Informally, functions of mixed monotonicity are those that are nondecreasing in some dimensions of the player's types and possibly nonincreasing in other dimensions.

[2] Kagel (1995) and Kagel and Levin (2015) are valuable surveys on the ongoing experimental work on auctions.

most of the empirical literature focusing on the case of identical goods (multi-unit auctions).[3] One of the main hurdles to progress beyond these settings is the lack of development of the theory. Thus extending the class of games for which we can characterize pure-strategy equilibria is a necessary step towards new developments in the empirical analysis of data generated by auctions. Finally, albeit beyond the scope of the present paper, a more general equilibrium existence result is of interest from the point of view of normative economics. By allowing for a richer strategic environment, our result can lead to policy questions that have not been considered before, such as how to auction strictly substitute goods.

To illustrate the scope of our main result, we study the equilibrium properties in a class of first-price auctions and a class of all-pay auctions. The applications to auctions we present here are parsimonious and intuitive. Their simplicity is a consequence of the breadth and flexibility of our main result. While these quasi-supermodularity and single-crossing assumptions are natural in some settings, there are many economic situations in which these assumptions entail unreasonable restrictions. The three auctions we describe illustrate ubiquitous economic environments in which the complementarity assumptions fail for various natural reasons.

The first application is an all-pay auction model in which bidders have one-dimensional type and action spaces, interdependent valuations, and correlated types in ways that may fail the monotone likelihood ratio property. Nevertheless, we are able to show that, under a condition more general than the weak monotonicity condition of Siegel (2014), this auction has an equilibrium in pure strategies that are of bounded total variation.

The second application is a first-price auction in which bidders' types are multidimensional and their valuations are interdependent, although restricted to be of polynomial form. Since polynomial functions are dense in the set of measurable functions, this auction demonstrates how our main result can be applied to show existence of equilibrium in models that are very close to games in which players have arbitrary, unrestricted payoff functions.

The remainder of the paper has the following structure. The mathematical framework, which concerns absolute retracts and ordered spaces, is described in Section 2. The class of Bayesian games our result covers is described formally in Section 3, where the main existence result is proved. Section 4 describes two auctions that illustrate the flexibility and scope of our result. Section 5 discusses sufficient conditions on the

[3] Athey and Haile (2007) provide an excellent survey of structural econometric approaches to auctions.

primitives of the game; these sufficient conditions, albeit stronger, require no preparation or mathematical preliminaries. Section 6 explains how the main results in Athey (2001), McAdams (2003), and Reny (2011) can be derived as a consequence of our result. The Appendix contains most proofs.

2. Mathematical framework

We now review the basic mathematical frameworks that are combined to yield the results in this paper: absolute retracts, lattice theory, and abstract simplicial complexes.

2.1. Absolute retracts.

Fix a metric space X. If Y is a metric space, a set $Z \subseteq Y$ is a *retract* of Y if there is a continuous function $r\colon Y \to Z$ with $r(z) = z$ for all $z \in Z$. Such function r is called a *retraction*. The space X is an *absolute retract*[4] (AR) or an *absolute neighborhood retract* (ANR) if, whenever X is homeomorphic to a closed subset Z of a metric space Y, Z is a retract of Y or a retract of a neighborhood of itself, respectively. Since the "is a retract of" relation is transitive, a consequence is that a retract of an AR (ANR) is an AR (ANR). An ANR is an AR if and only if it is contractible (Borsuk, 1967, Theorem 9.1). A *contractible* set is a set that can be reduced to one of its points by a continuous deformation. Formally, a set X is said to be contractible if it is homotopic to one of its points $x \in X$, that is, if there is a continuous map $h\colon [0,1] \times X \to X$ such that $h(0, \cdot\,)\colon X \to X$ is the identity map and $h(1, \cdot\,)\colon X \to X$ is the constant map sending each point to x. In this case, the mapping h is denoted a *contraction*.

The Eilenberg-Montgomery fixed point theorem (Eilenberg and Montgomery, 1946) asserts that if X is a nonempty compact AR, $F\colon X \twoheadrightarrow X$ is a closed-graph correspondence, and the values of F are "acyclic," then F has a fixed point. For the purposes of this paper, it suffices to know that a contractible set is acyclic, so that F has a fixed point if its values are contractible. Kinoshita (1953) gives an example of a compact contractible subset of $\mathbb{R}^3$ and a continuous function from this space to itself that does not have a fixed point, so the assumption that X is a compact AR cannot be weakened to "compact and contractible."

In Athey (2001) and McAdams (2003) a large part of the analytic effort is devoted to showing that the set of monotone best responses to a profile of monotone

[4]The terms "metric absolute retract" and "absolute retract for metric spaces" are used in mathematical literature that also considers spaces that satisfy the embedding condition for other types of topological spaces.

strategies is convex valued. However, Reny (2011) provides a simple construction that shows that this set is contractible valued. In addition, passing to the more general Eilenberg-Montgomery fixed point theorem allows many of the assumptions of earlier results to be relaxed. The weakening of hypotheses does not complicate the proof of contractibility, but instead there is the challenge of showing that the set of (equivalence classes of) monotone pure strategy profiles is an AR. Since the set of monotone strategy profiles is contractible, Reny could demonstrate this by verifying the sufficient conditions for a space to be an ANR given by Theorem 3.4 of Dugundji (1965), which is derived from necessary and sufficient conditions given earlier in that paper that in turn build on Dugundji (1952) and Dugundji (1957).

A central theme of this paper is that there is a variety of conditions that imply that a space is an AR. Any of these is potentially the basis of an equilibrium existence result for some type of Bayesian game, and we will provide novel existence results of this sort. In particular, it will be possible to verify other sufficient conditions for a space to be an AR that are related to the order structure of the space of monotonic strategy profiles, and are thus in a sense more natural. Perhaps more importantly, they are flexible, allowing for existence under different hypotheses.

2.2. **Simplicial complexes.** An *abstract simplicial complex* is a pair $\Delta = (X, \mathcal{X})$ in which X is a set of *vertices* and $\mathcal{X}$ is a collection of finite subsets of X that contains every subset of each of its elements. Elements of $\mathcal{X}$ are called *simplexes*. The *realization* of Δ is

$$|\Delta| = \left\{ \pi \in \mathbb{R}^X_+ \colon \sum_{x \in X} \pi_x = 1, \text{ and } \operatorname{supp} \pi \in \mathcal{X} \right\},$$

where $\operatorname{supp} \pi = \{x \in X \colon \pi_x > 0\}$. For a simplex $Y \in \mathcal{X}$, let $|Y| = \{\pi \in |\Delta| \colon \operatorname{supp} \pi \in Y\}$. Then $|\Delta| = \bigcup_{Y \in \mathcal{X}} |Y|$. We will always assume that $\{x\} \in \mathcal{X}$ for every $x \in X$. We endow $|\Delta|$ with the *CW topology*, which is the topology in which each $|Y|$ has its usual topology and a set is open whenever its intersection with each $|Y|$ is open.

Let Z be a topological space. A correspondence $F \colon \mathcal{X} \setminus \{\emptyset\} \to Z$ is a *contractible carrier* that sends simplexes of Δ to subsets of Z if, for every nonempty $Y \in \mathcal{X}$:

(a) $F(Y)$ is contractible, and
(b) if $\emptyset \neq Y' \subseteq Y$, then $F(Y') \subseteq F(Y)$.

Moreover, a continuous function $f \colon |\Delta| \to Z$ is *carried by* F if $f(|Y|) \subseteq F(Y)$ for every $Y \in \mathcal{X}$. The following result is from Walker (1981).

Lemma 2.1 (Walker's carrier lemma)**.** *If F is a contractible carrier from Δ to Z, then there is a continuous function $f\colon |\Delta| \to Z$ carried by F, and any two such functions are homotopic.*

For the remainder of the paper, we reserve the notation Δ for the abstract simplicial complex in which $\mathcal{X}$ is the collection of all finite subsets of X.

2.3. **Posets and semilattices.** A *partially ordered set (poset)* is a set X endowed with a binary relation $\leq$ that is reflexive ($x \leq x$ for every x), transitive, and anti-symmetric ($x \leq y$ and $y \leq x$ implies $x = y$). Let

$$G_{\leq} = \{\, (x, y) \in X \times X : x \leq y \,\}.$$

If X is endowed with a σ-algebra Σ, the partial order $\leq$ is said to be *measurable* if $G_{\leq}$ is an element of the product σ-algebra $\Sigma \otimes \Sigma$. If X is endowed with a topology, the partial order $\leq$ is said to be *closed* if $G_{\leq}$ is closed in the product topology of $X \times X$. If X is a subset of a real vector space, the partial order $\leq$ is said to be *convex* if $G_{\leq}$ is convex. Since $\{\, (x, x) : x \in X \,\} \subseteq G_{\leq}$, if $\leq$ is convex, then X is necessarily convex.

A partially ordered set X is a *semilattice*[5] if any two elements $x, y \in X$ have a least upper bound $x \vee y$. If this is the case, then the semilattice operation is obviously associative, commutative, and idempotent. That is, $x \vee x = x$ for all $x \in X$. Conversely, if $\vee$ is a binary operation on X that is associative, commutative, and idempotent, then there is a partial order on X given by $x \leq y$ if and only if $x \vee y = y$ that makes X a semilattice for which $\vee$ is the least upper bound operator.[6] If the greatest lower bound of any two elements $x, y \in X$ exists, then it is denoted by $x \wedge y$.

A subset $Y \subseteq X$ is a *subsemilattice* if $x \vee y \in Y$ for all $x, y \in X$. Evidently the intersection of any collection of subsemilattices is a subsemilattice. A *metric semilattice* is a semilattice endowed with a metric such that $(x, y) \mapsto x \vee y$ is a continuous function from $X \times X$ to X. A metric semilattice is *locally complete* if, for every $x \in X$ and every neighborhood U of x, there is a neighborhood W such that every nonempty $W' \subseteq W$ has a least upper bound that is contained in U.

2.4. **The hyperspace of a compact metric semilattice.** If X is a compact metric space, the *hyperspace* of X is the set $\mathcal{S}(X)$ of nonempty closed subsets of X endowed

[5]This concept is often described as a *join semilattice* in contexts in which one also considers meet semilattices, which are posets in which any pair of elements has greatest lower bound.

[6]Verification of the details underlying this assertion is straightforward.

with the topology that has as a subbasis the set of sets of the form

$$N(U, V) = \{ C \in \mathcal{S}(X) \colon C \subseteq U \text{ and } C \cap V \neq \emptyset \}$$

where $U, V \subseteq X$ are open. The space X is *locally connected* if it has a base of connected open sets. Wojdysławski (1939) showed that if X is connected and locally connected, then $\mathcal{S}(X)$ is an AR. (Kelley (1942) reproves this result, and places it in a broader context.)

Now suppose X is a compact metric semilattice. It is easy to show that any subset $S \subseteq X$ has a least upper bound that we denote by $\vee S$. We say that X *has small subsemilattices* if it has a neighborhood base of subsemilattices, which is called an *idempotent basis*. It is easy to show that X is locally complete if and only if it has small subsemilattices. Identifying each $x \in X$ with $\{x\} \in \mathcal{S}(X)$, we may regard X as a subset of $\mathcal{S}(X)$. McWaters (1969) showed that if X has small subsemilattices, then the map $C \mapsto \vee C$ is continuous and consequently a retraction. As McWaters points out, in conjunction with Wojdysławski's result, this result implies the following theorem.

Theorem 2.2. *If X is connected, locally connected, and locally complete, then it is an AR.*

In the Bayesian game considered in Reny (2011), type and action spaces are assumed to be semilattices, and strategy spaces are thus ordered by the induced pointwise ordering. As a result, the subset of monotone strategies is a sub-semilattice, therefore contractible to its least upper bound. In the following section, we extend this result to more general partially ordered subsets of strategies, including subsets that are not necessarily given the induced pointwise ordering or that may not have a least upper bound.

2.5. **A new class of retracts.** We can now describe a new class of absolute retracts, generated by combining the order structure of posets and abstract simplicial complexes. Let X be a metric space and a poset. (We do not assume that the order is closed.) A (finite) *chain* in X is a (finite) completely ordered subset of X. When X is a partially ordered space, we consider the *order complex* $\Gamma = (X, \mathcal{X}^\Gamma)$ of X. The order complex Γ is the abstract simplicial complex for which the set of vertices is X itself and the collection of simplexes $\mathcal{X}^\Gamma$ is the collection of finite chains of X. If $\Gamma = (X, \mathcal{X}^\Gamma)$ is the order complex of X and $\Delta = (X, \mathcal{X})$ is the abstract simplicial complex in which the simplexes are all finite subsets of X, then $\mathcal{X}^\Gamma \subseteq \mathcal{X}$, and we regard the geometric realization $|\Gamma|$ as a subspace of $|\Delta|$. If Y is a finite subset of X, then $Y \in \mathcal{X}$ and we denote by $|Y^\Gamma|$ the realization of Y on the order complex Γ, that is, $|Y^\Gamma| = |Y| \cap |\Gamma|$.

The following definition describes a novel mathematical concept.[7] We say that a sequence of subsets of X *converges* to $x \in X$ if the sequence is eventually contained in each neighborhood of x.

Definition 2.3. A *hulling* of X is a collection $\mathcal{H}$ of subsets of X such that:

(a) $\mathcal{H}$ is closed under intersection;
(b) every finite subset of X is contained in some element of $\mathcal{H}$;
(c) for each nonempty $Y \in \mathcal{H}$, the realization $|Y^\Gamma|$ is contractible.

When Y is a finite subset of X, the *$\mathcal{H}$-hull* of Y, denoted by $\mathcal{H}(Y)$, is the intersection of all $Y' \in \mathcal{H}$ containing Y. The hulling $\mathcal{H}$ is *small* if, for any sequence of finite sets Y_n converging to a point x, the sequence $\mathcal{H}(Y_n)$ also converges to x.

Note that if X has an upper bound, then $|\Gamma|$ is contractible. Therefore, if X is a semilattice and $\mathcal{H}$ is the collection of all finite sub-semilattices of X, then $\mathcal{H}$ is a hulling. Figure 1 shows examples of other kinds of sets that can compose a hulling. Figure 2 is an example of a set Y for which $|\mathcal{H}(Y)^\Gamma|$ is not contractible, and thus cannot belong to a hulling.

Definition 2.4. A *monotone realization* is a continuous function $h\colon |\Gamma| \to X$. A monotone realization h is said to be *local* whenever, for every sequence Y_n of nonempty finite chains converging to $x \in X$, the sequence $h(|Y_n|)$ also converges to x.

Together, the notions of hulling and monotone realization describe what we call order-convexity.

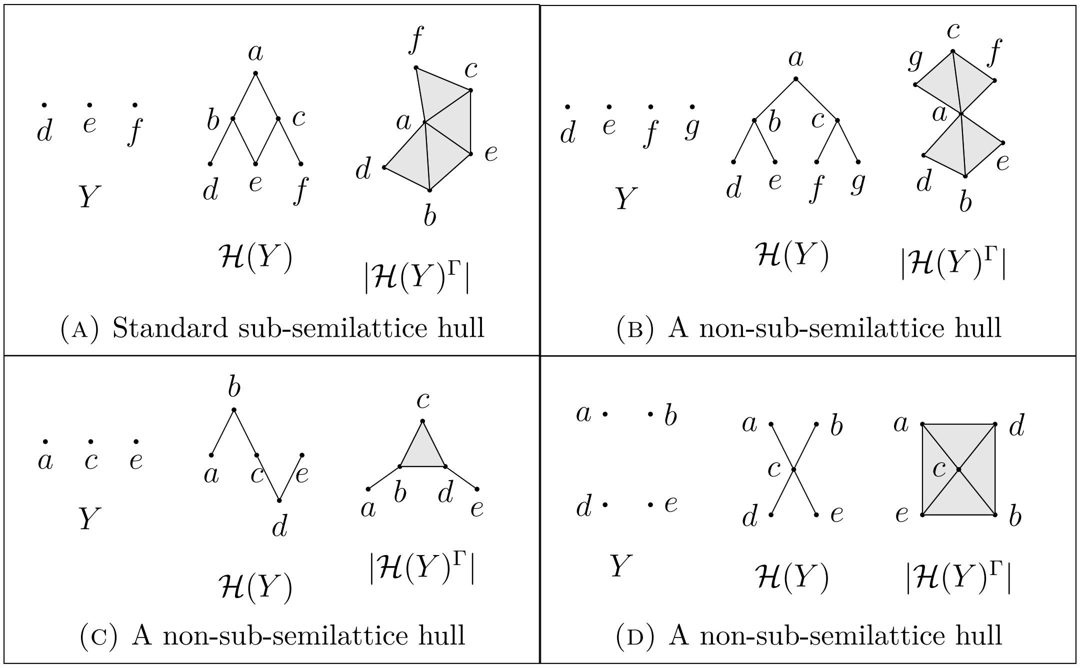
Definition 2.5. A partially ordered set $(X, \leq)$ is *order-convex* if there is a small hulling $\mathcal{H}$ and a local monotone realization h for X such that

(a) for every finite subset $Y \subseteq X$, we have $\mathcal{H}(Y) \subseteq X$; and
(b) for every finite chain Y in X, we have $h(|\mathcal{H}(Y)^\Gamma|) \subseteq X$.

Remark 2.6. Section B in the Appendix proposes easy-to-check conditions for a hulling to be small and a monotone realization to be local.

The following lemma establishes that every order-convex, separable, metric space is an absolute retract. Lemma 2.7 is the main tool used to prove the results.

[7]It is a generalization of the notion of a *zellij* in McLennan, Monteiro, and Tourky (2011).



(A) Standard sub-semilattice hull

(B) A non-sub-semilattice hull

(C) A non-sub-semilattice hull

(D) A non-sub-semilattice hull

FIGURE 1. Examples of sets $\mathcal{H}(Y) \in \mathcal{H}$

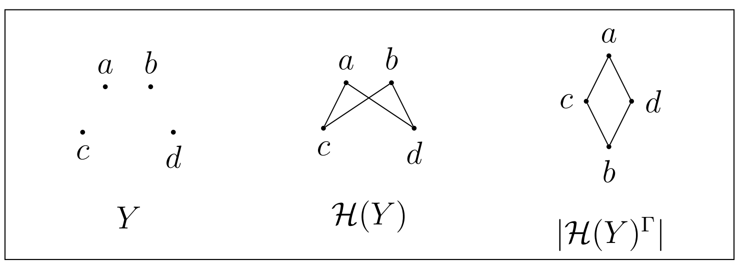


FIGURE 2. Example of a set that cannot compose a hulling

Lemma 2.7. *If $(X, \leq)$ is partially ordered space that is separable, metric, closed, and order-convex, then X is an absolute retract.*

The proof of Lemma 2.7 can be found in the Appendix A.

3. Class of Bayesian games

We consider the class of Bayesian games described by the following tuple

$$G = ((T, \mathcal{T}), \pi, A, u) .$$

The space $(T, \mathcal{T}) = \otimes_i (T_i, \mathcal{T}_i)$ is a product of N measurable spaces of *types*. The probability measure $\pi \in \Delta(T)$ is the *common prior*; we let π_i be the marginal of π on T_i. The space $(A, \mathcal{A}) = \otimes_i (A_i, \mathcal{A}_i)$ is a product of N measurable spaces of *actions*;

we assume that each A_i is a compact subset of some Banach space L_i and is endowed with a σ-algebra $\mathcal{A}_i$ that includes the Borel sets. Finally, the tuple $u = (u_1, \dots, u_N)$ is a profile of bounded jointly measurable *payoff functions* $u_i \colon T \times A \to \mathbb{R}$.[8]

A (pure) *strategy* for player i is a function from T_i to A_i that is π_i-a.e. equal to a measurable function. Let S_i be the set of player i's strategies, and let $S = \prod_i S_i$ be the set of *strategy profiles*. We regard the space of strategies S_i as a subspace of $L^1(T_i, \pi_i)$, the space of Bochner-integrable functions (equivalence classes) from T_i to L_i, with the L^1-norm topology. For each $s \in S$ and each i, player i's *expected payoff* is

$$U_i(s) = \int_T u_i(t, s(t)) \, d\pi(t).$$

A strategy $s_i \in S_i$ is a *best response* to $s_{-i} \in S_{-i}$ if $U_i(s_i, s_{-i}) \geq U_i(s'_i, s_{-i})$ for all $s'_i \in S_i$. A strategy profile $s \in S$ is an *equilibrium* if, for each i, s_i is a best response to s_{-i}.

Let $B_i \colon S_{-i} \to S_i$ denote the best response correspondence of player i:

$$B_i(s_{-i}) = \left\{ s_i \in S_i \colon s_i \in \arg\max_{s_i \in S_i} U_i(s_i, s_{-i}) \right\}.$$

Let $B \colon S \to S$ be the cartesian product of the B_i: $B(s) = B_1(s_{-1}) \times \cdots \times B_N(s_{-N})$.

We make the following assumption on the common prior.

Assumption A.1. *For every player i, the common prior π is absolutely continuous with respect to the product of its marginals.*[9]

We also make the following assumption on the players' payoffs.

Assumption A.2. *For every player i, the function $u_i \colon T \times A \to \mathbb{R}$ is continuous in a and measurable in t.*

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[8]We use standard notation for the indexing of player profiles: for a N-tuple $(X_i)_{i=1}^N$ of sets we let $X = \prod_i X_i$, and for each player i we let $X_{-i} = \prod_{j \neq i} X_j$. Vectors in X are called *profiles*. A profile $x \in X$ is also written as (x_i, x_{-i}) where x_i is the i-th coordinate of x and x_{-i} is the projection of x into X_{-i}. We also use standard notation for probability: if (X, Σ) is a measurable space, then $\Delta(X)$ is the set of probability measures on X.

[9]If the marginals are purely atomic, then this assumption is trivially true. If the marginals are nonatomic, then an application of the Radon-Nikodym theorem yields that each player i's *ex ante* payoff U_i can be written as the integral of their *interim* payoff V_i. Therefore, if $s_i, s'_i \in B_i(s_{-i})$ are two best responses, then the piecewise combination $s_i \mathbf{1}_E + s'_i \mathbf{1}_{T_i \setminus E} \in S_i$ is also an element of $B_i(s_{-i})$ for every measurable set $E \subseteq T_i$. Further, a fixed point of the best response correspondence B is a Bayesian-Nash equilibrium in the traditional *(interim)* sense.

Under Assumptions A.1 and A.2, the best response correspondence B is non-empty and has closed graph by an application of the Vector Dominated Convergence Theorem and Berge Maximum Theorem.[10] We are ready to state our main result.

Theorem 3.1. *Suppose that Assumptions A.1–A.2 are satisfied. If, for every player i, there is a compact, order-convex subset of strategies $K_i \subseteq S_i$ such that $B_i(s_{-i}) \cap K_i$ is a nonempty, order-convex set for every $s_{-i} \in K_{-i}$, then the game G has an equilibrium in K.*

Proof. By Lemma 2.7, every compact, order-convex subset of strategies is an absolute retract. Consider the subcorrespondence of best responses $\overline{B}\colon K \twoheadrightarrow K$, given by

$$\overline{B}(s) = B(s) \cap K\,.$$

As defined, $\overline{B}$ has closed-graph, and compact, order-convex values. Therefore, it satisfies the hypotheses of Eilenberg-Montgomery fixed point theorem. Hence it has a fixed point in K, which is a Bayesian equilibrium of the game G. □

Remark 3.2. Theorem 3.1 not only helps proving existence of equilibrium results, but it also provides additional, useful information regarding how the equilibrium found looks like. In fact, this is the main motivation for the analysis in Athey (2001), McAdams (2003), and Reny (2011).

Notice that this result does not require the players' type and action spaces to be partially ordered. Nor it requires the partial order on K_i to be induced by the pointwise order. In fact, it allows for partial orders that may depend on the whole strategy, as a function from types to actions. Further, Theorem 3.1 does not require the marginals of the probability measure π to be atomless. It is, however, easier to get order-convex best responses when the priors are atomless, as all three auctions analyzed in Section 4 show.

4. Applications to auctions

We present two applications of the main result to auctions. Most of the auction literature relies on existence of monotone equilibrium.[11] Although it is not difficult to write auctions in which monotonicity fails, as the examples in Jackson (2009), Reny and Zamir (2004), and McAdams (2007) show, it remains unclear whether or not

[10]We refer the reader to Aliprantis and Border (2006, Theorems 11.46, 13.6, 17.11, 17.28, and 17.31).
[11]We refer to Kaplan and Zamir (2015) and Klemperer (1999) for excellent surveys.

non-monotonicities in the best-response correspondence pose a serious threat to the existence of equilibrium. The auctions in this section shed some light on this issue.

The first example concerns an all-pay auction that encompasses and generalizes some standard existence results for such settings. The main advance here is in allowing for interdependent valuations and information structures that may fail the weak single-crossing property, yielding equilibria that are not necessarily monotone in players' types. This all-pay auction shows that, even when restricted to the class of games with unidimensional type and action spaces, Theorem 3.1 extends the analysis of pure-strategy equilibria to a broader range of models. The second application involves a first-price auction in which bidders' types are multidimensional and bidders' valuations can be arbitrarily interdependent. The purpose of the second application is to highlight that, within the class of auctions with multidimensional types, Theorem 3.1 allows for an analysis of a much richer class of bidders' preferences. The proofs of all claims made in this section are in Appendix C.

Before describing the auctions, we make a closing remark with regards to modelling choices. In all of the applications in this paper, bidders submit bids at predetermined discrete levels, that is, there exists a minimal increment by which the bid may be raised. Although the auction literature deals almost entirely with continuous bids, in practice bidders are not able to choose their bid from a continuum. At best, the smallest currency unit imposes such restriction on feasible bids; at worst, the auctioneer may restrict the set of acceptable bids even further. We thus consider this a natural assumption, which yields a model that is both parsimonious and realistic. However, it is possible to extend the analysis in this section to permit a continuum bids under additional assumptions.

4.1. **All-pay auction.** Consider an all-pay auction with incomplete information. After observing the realization of their signals, bidders submit their bids, and pay their bids regardless of whether or not they win the object. This kind of model has been used to investigate rent-seeking and lobbying activities, competitions for a monopoly position, competitions for multiple prizes, political contests, promotions in labor markets, trade wars, and R&D races with irreversible investments.

There is a single object for sale and I bidders. Each bidder i observes the realization of a private signal $t_i \in [\underline{\tau}, \overline{\tau}] = T_i$. Signals of all bidders $T = (T_1, \ldots, T_I)$ are drawn from some joint distribution with density $f\colon [\underline{\tau}, \overline{\tau}]^I \to \mathbb{R}_+$. The value of the object being auctioned to bidder i is given by the measurable mapping $v_i\colon [\underline{\tau}, \overline{\tau}]^I \to$

$\mathbb{R}$. We make the following assumption on the primitives of the model, which is a generalization of the weak monotonicity condition of Siegel (2014).

Assumption B.1. *For each bidder i, there is a finite partition of the set of signals $T_i = \cup_n \mathcal{I}_i^n$ into subintervals $\mathcal{I}_i^n$ such that for every t_{-i} the restriction of the weighted valuation $v_i(t_i, t_{-i}) f(t_{-i} \mid t_i)$ to each subinterval $\mathcal{I}_i^n$ is monotone[12] in t_i.*

Remark 4.1. Essentially, Assumption B.1 puts an upper bound on the number of times bidder i's weighted valuation changes direction, it allows for very general interdependence and correlation structures. In particular, it allows for the weighted valuations to be nondecreasing on some subintervals and nonincreasing on others, and does not impose any restrictions across the subintervals $\{\mathcal{I}_i^n\}_n$. The independent private value auction corresponds to the special case in which $v_i(t_i, t_{-i}) = t_i$ and $f(t_{-i} \mid t_i)$ does not depend on t_i.

Most of the literature on all-pay auctions concentrates on the case in which the players' weighted valuation functions are nondecreasing, yielding monotone equilibria. Assumption B.1 is a natural generalization of that single-crossing condition.

Given signal t_i, bidder i places a bid b, chosen from a finite set of bids $\mathcal{B} \subseteq \mathbb{R}$. The allocation of prizes is determined by the profile of bids. In particular, we assume that there is a function $\alpha\colon \{1, \ldots, I\} \times \mathcal{B}^I \to [0, 1]$, such that $\alpha(b)$ is a probability measure over bidders. The interpretation is that $\alpha_i(b)$ is the probability that bidder i gets the object, given profile of bids b. We only assume that the allocation mapping $b_i \mapsto \alpha_i(b)$ is nondecreasing, that is, a higher bid will increase the probability that bidder i gets the object.

A strategy for bidder i is a measurable function $\beta_i\colon T_i \to \mathcal{B}$. Given a profile of strategies of other bidders β_{-i}, bidder i's interim payoff is given by

$$V_i(b \mid t_i, \beta_{-i}) = \int_{[\underline{\tau}, \overline{\tau}]^{I-1}} \alpha_i(b, \beta_{-i}(t_{-i})) v_i(t) f(t_{-i} \mid t_i)\, dt_{-i} - b\,.$$

Given a profile of strategies for all bidders β, bidder i's ex ante payoff is then given by

$$U_i(\beta) = \int_{[\underline{\tau}, \overline{\tau}]} V_i(\beta_i(t_i) \mid t_i, \beta_{-i}) f(t_i)\, dt_i\,.$$

Theorem 3.1 implies that this auction has a Bayesian-Nash equilibrium in which each bidder i uses a strategy that is monotone in t_i when restricted to each subinterval $\mathcal{I}^n$.

[12]By monotone, we mean either nonincreasing or nondecreasing.

4.2. **First-price auction with interdependent values.** Consider a sealed-bid first-price auction in which bidders' types are multidimensional and possibly interdependent. This kind of model has been used to study, for example, procurement auctions, in which bidders are suppliers who try to underbid each other to sell an object or provide a service to a potential buyer. Government contracts are usually awarded by procurement auctions, and firms often use this auction format when buying inputs or subcontracting work.

There is a single object for sale and N bidders. Each bidder i's type is a vector $t_i = (t_{i1}, \ldots, t_{iK}) \in [\underline{\tau}, \overline{\tau}]^K$. Bidders' types are independently drawn. Let $f_i \colon [\underline{\tau}, \overline{\tau}]^K \to \mathbb{R}_+$ denote the density distribution of bidder i's types. The value of the object being auctioned to bidder i is given by the measurable map $v_i \colon [\underline{\tau}, \overline{\tau}]^{KN} \to \mathbb{R}_+$.

We assume that the map v_i is the sum of polynomial functions in each bidders' vector of types. More precisely, bidder i's valuation function can be written as

$$v_i(t) = \sum_{j=1}^{N} \sum_{m \in M_j} \alpha_m t_{j1}^{d_1^m} \cdots t_{jK}^{d_K^m} ,$$

where M_j is a finite index set for each $j = 1, \ldots, N$ and, for each $m \in M_j$, the number α_m is the coefficient of the m-th term and d_k^m are nonnegative integers.

The interpretation is that each dimension k of bidder i's type represents an inherent characteristic of the object, and bidders observe a noisy and independent informative signal regarding these characteristics. Each of these characteristics may or may not be intrinsically desirable. Thus, while we do not rule out symmetric bidders, we do allow for heterogeneous preferences in the sense that different bidders feel differently about each characteristic. In particular, for each dimension k, it may be the case that some bidders prefer higher levels of k, whereas other bidders may prefer lower or even intermediate levels.

Bidder i observes the realization of her private type t_i, that gives information about the characteristics of the object. Upon observing t_i, bidder i submits a bid b_i from a finite set of bids $\mathcal{B} \subseteq \mathbb{R}$. Given a vector $b = (b_1, \ldots, b_N)$ of bids of all bidders, the object is awarded to the highest bidder, who pays her bid. If there is a tie at the highest bid, then the object is awarded to one of the highest bidders with equal probability. Let $\rho_i(b) \in [0, 1]$ denote the probability that bidder i gets the object given profile of bids b. Given a vector b of bids, bidder i's payoff is given by

$$u_i(b; t) = \rho_i(b)[v_i(t) - b_i] .$$

In this context, a strategy for bidder i is a measurable function $\beta_i\colon [\underline{\tau}, \overline{\tau}]^K \to \mathcal{B}$. Given a profile of strategies for all bidders β, bidder i's *ex ante* payoff is given by

$$U_i(\beta) = \int_{[\underline{\tau},\overline{\tau}]^{NK}} u_i(\beta(t); t) f_1(t_1) \dots f_N(t_N) \, \mathrm{d}\, t \, .$$

Theorem 3.1 implies that this auction has a Bayesian-Nash equilibrium in which each bidder i uses a strategy that is (locally) nondecreasing in t_{ik} whenever $\frac{\partial v_i}{\partial t_{ik}}(t) \geq 0$, and (locally) nonincreasing whenever $\frac{\partial v_i}{\partial t_{ik}}(t) \leq 0$.

5. Sufficient conditions on primitives

For readers interested in applications, it may be easier to verify sufficient, but less general, conditions that lead to the existence of Bayesian equilibria. Here we provide two sets of such conditions. The first set, formally stated in Theorem 5.1, is written in terms of payoff differences. The second set of conditions, stated in Corollary 5.3, imposes restrictions on differentiable payoffs.

Throughout this section, we make the following assumptions:

(1) Each player i's type space $T_i = [\underline{\tau}_i, \overline{\tau}_i]^{M_i} \times [\underline{\tau}_i, \overline{\tau}_i]^{M_i'}$ is a nondegenerate Euclidean cube, with the coordinate-wise partial order.
(2) Each player i's types are distributed according to the probability density f_i over T_i, not necessarily everywhere positive, but independently of other players' types.
(3) Each player i's set of actions $A_i = [\underline{\alpha}_i, \overline{\alpha}_i]^{N_i} \times [\underline{\alpha}_i, \overline{\alpha}_i]^{N_i'}$ is an Euclidean cube, endowed with the coordinate-wise partial order.
(4) Each player i's payoff function $u_i\colon A \times T \to \mathbb{R}$ is bounded, measurable in t, and continuous in a.

The following theorem establishes that it suffices to check whether the first differences of payoffs are increasing in specific directions of the players' action and type spaces.

Theorem 5.1. *Suppose that the payoff function $u_i\colon A \times T \to \mathbb{R}$ of every player i satisfies the following two conditions:*

(a) *Take any given $a_{-i} \in A_{-i}$ and $t \in T$. If $a, a' \in A_i$, then*

$$\begin{aligned} &u_i(a, a_{-i}; t) - u_i((a_{N_i} \wedge a'_{N_i}, a_{N_i'} \vee a'_{N_i'}), a_{-i}; t) \geq 0 \\ \Rightarrow \qquad &u_i((a_{N_i} \vee a'_{N_i}, a_{N_i'} \wedge a'_{N_i'}), a_{-i}; t) - u_i(a', a_{-i}; t) \geq 0 \, . \end{aligned}$$

(b) *Take any given* $a_{-i} \in A_{-i}$ *and* $t_{-i} \in T_{-i}$*. Suppose* $a_i, a'_i \in A_i$ *are such that* $a_{ik} \geq a'_{ik}$ *for* $k \in N_i$ *and* $a_{ik} \leq a'_{ik}$ *for* $k \in N'_i$*; and* $t_i, t'_i \in T_i$ *are such that* $t_{i\ell} \geq t'_{i\ell}$ *for* $\ell \in M_i$ *and* $t_{i\ell} \leq t'_{i\ell}$ *for* $\ell \in M'_i$*, then*

$$\begin{aligned} & u_i(a_i, a_{-i}; t'_i, t_{-i}) - u_i(a'_i, a_{-i}; t'_i, t_{-i}) \geq 0 \\ \Rightarrow \qquad & u_i(a_i, a_{-i}; t_i, t_{-i}) - u_i(a'_i, a_{-i}; t_i, t_{-i}) \geq 0 \,. \end{aligned}$$

Then there exists a Bayesian equilibrium in which each player i *plays a pure strategy* s_i *such that the projection* $t_i \mapsto s_{ik}(t_i)$*, with* $k \in N_i$ *(*$k \in N'_i$*), is nondecreasing (nonincreasing) in* $t_{i\ell}$ *for* $\ell \in M_i$ *and nonincreasing (nondecreasing) in* $t_{i\ell}$ *for* $\ell \in M'_i$*.*

Remark 5.2. In Theorem 5.1, if $N'_i = \emptyset$ and $M'_i = \emptyset$, then condition (a) reduces to the usual quasi-supermodularity of McAdams (2003) and condition (b) reduces to the single-crossing property of McAdams (2003). Then the theorem implies existence of a monotone Bayesian equilibrium.

We can now state a corollary of Theorem 5.1 for the case when the players' payoffs are twice continuously differentiable.

Corollary 5.3. *Suppose that for each player* i*, the payoff function* $u_i \colon A \times T \to \mathbb{R}$ *is twice continuously differentiable and satisfies the following two conditions:*

(a) *For every* $a \in A$ *and* $t \in T$*,*

$$\begin{aligned} \frac{\partial^2 u_i(a;t)}{\partial a_{ik} \partial a_{ik'}} \geq 0 \qquad & \text{if either } k, k' \in N_i \text{ or } k, k' \in N'_i \text{; and} \\ \frac{\partial^2 u_i(a;t)}{\partial a_{ik} \partial a_{ik'}} \leq 0 \qquad & \text{if } k \in N_i \text{ and } k' \in N'_i \,. \end{aligned}$$

(b) *For every* $a \in A$ *and* $t \in T$*,*

$$\begin{aligned} \frac{\partial^2 u_i(a;t)}{\partial a_{ik} \partial t_{i\ell}} \geq 0 \qquad & \text{if } k \in N_i \text{ and } \ell \in M_i \text{; or } k \in N'_i \text{ and } \ell \in M'_i \text{;} \\ \frac{\partial^2 u_i(a;t)}{\partial a_{ik} \partial t_{i\ell}} \leq 0 \qquad & \text{if } k \in N'_i \text{ and } \ell \in M_i \text{; or } k \in N_i \text{ and } \ell \in M'_i \,. \end{aligned}$$

Then there exists a Bayesian equilibrium in which each player i *plays a pure strategy* s_i *such that the projection* $t_i \mapsto s_{ik}(t_i)$*, with* $k \in N_i$ *(*$k \in N'_i$*), is nondecreasing (nonincreasing) in* $t_{i\ell}$ *for* $\ell \in M_i$ *and nonincreasing (nondecreasing) in* $t_{i\ell}$ *for* $\ell \in M'_i$*.*

All proofs for this section can be found in Appendix D.

6. Literature

There is an extensive literature concerned with existence of pure-strategy equilibrium for Bayesian games with a continuum of types and general action spaces, pioneered by Milgrom and Weber (1985).[13] In games where the strategy spaces are function spaces, such as Bayesian games, the existence question is often inextricably tied to the characterization question: do equilibrium strategies possess some prescribed structure of interest? Within several economic settings, it is natural to look in particular for equilibria in which each agent follows a pure strategy that is an increasing function of her type. Topkis (1979) was the first to develop a general theory and method for this kind of analysis. In the context of Bayesian games, Athey (2001), McAdams (2003), van Zandt and Vives (2007), van Zandt (2010), Reny (2011), and Amir and Lazzati (2016) all provide results of this sort. Prokopovych and Yannelis (2019, 2022) provide existence results for general auctions and contests.[14] Remarkably, Reny (2011) introduces far-reaching new techniques applying the fixed point theorem of Eilenberg and Montgomery (1946, Theorem 5). This is done by showing that with atomless type spaces the set of monotone functions is an absolute retract and when the values of the best response correspondence are non-empty sub-semilattices of monotone functions, they too are absolute retracts. This paper extends this line of research, providing a theory that encompasses Reny's results while generalizing the relevant methods. In this section, we show how the main result in Reny (2011), which generalizes Athey (2001) and McAdams (2003), can be derived from Theorem 3.1. As a reminder, we state Reny's main result in a concise form.

Theorem 6.1 (Theorem 4.1 of Reny (2011))**.** *Suppose that the following assumptions hold.*

(1) *For each player i,*
   (a) *π_i is atomless;*
   (b) *T_i is endowed with a measurable partial order for which there is a countable set $T_i^0 \subseteq T_i$ such that for every $E \in \mathcal{T}_i$ with $\pi_i(E) > 0$ there are $t_i, t_i' \in E$ with $[t_i, t_i'] \cap T_i^0 \neq \emptyset$;*
   (c) *A_i is compact metric space, and a semilattice with closed partial order;*
   (d) *either:*

[13]The most general result of existence of equilibrium in behavioral strategies is Balder (1988).
[14]Stochastic games form another class of games that shares these characteristics. Amir (1996) and Curtat (1996) establish existence of Markov-perfect equilibrium in strategies that are monotone in the state variable for a class of submodular games.

(i) *A_i is a convex subset of a locally convex topological vector space and the partial order on A_i is convex, or*
(ii) *A_i is a locally complete metric semilattice;*

(e) *$u_i(t, \cdot\,)$ is continuous for every $t \in T$.*

(2) *Each player's set of nondecreasing pure best responses is nonempty and closed with respect to the supremum operation whenever the other players use nondecreasing pure strategies.*

Then the Bayesian game has an equilibrium in nondecreasing pure strategies.

First, we show that, under the assumptions listed, the set of nondecreasing strategies is order-convex. Fix a player i. Being a compact, metric space, the set of actions A_i can be isometrically embedded in a Banach space L_i. The set of nondecreasing strategies M_i for player i is thus a subset of Bochner-integrable functions from T_i to L_i, under the L^1-norm topology. Partially order M_i according to the (almost everywhere) pointwise order, as follows

$$f_i \geq g_i \qquad \Longleftrightarrow \qquad f_i(t_i) \geq g_i(t_i) \qquad \pi_i - \text{a.e}\,.$$

Under this partial order, the set of nondecreasing strategies M_i is a metric semilattice. Further, by Reny (2011, Lemmas A.10 and A.11), the set M_i is L^1-norm compact. The next lemma establishes that M_i is also locally complete.

Lemma 6.2. *Under the assumptions of Theorem 6.1–(1), the set of nondecreasing strategies M_i for every player i is locally complete.*

Proof. **Case (1.d.i):** We show that, under assumption (1.c), if A_i is a convex subset of a locally convex topological vector space with a convex partial order, then A_i is a locally complete metric semilattice. Thus case (1.d.i) reduces to (1.d.ii). Given Reny (2011, Lemma A.18), it suffices to show that if a_n is a sequence of actions converging to a, then $b_m = \vee_{n \geq m} a_n$ also converges to a as m goes to infinity. Suppose b_m does not converge to a. Because A_i is compact, taking a subsequence if necessary, we may assume that b_m converges to $b \neq a$. Since $a_m \leq b_m$ for every m and $\leq$ is a closed order, it follows that $a \leq b$. And since $a \neq b$, it follows that $a < b$. Because A_i is a convex subset of a metric, locally convex topological vector space, with a closed order, there exist two disjoint, convex neighborhoods U of a and V of b such that $a' < b'$ for every $a \in U$ and $b \in V$. Pick $\alpha \in (0, 1)$ such that $\alpha a + (1 - \alpha)b \in V$. Since $\leq$ is closed, it follows that $\alpha a + (1 - \alpha)b < b$, and notice that there is a convex neighborhood W of b such that $\alpha a + (1 - \alpha)b < b'$ for every $b' \in W$. Let M be an integer such that $a_n \in U$ for every $n \geq M$ and $b_m \in W$ for every $m \geq M$. Therefore, $\alpha a + (1-\alpha)b$ is an upper bound on the set $\bigcup_{n \geq M}\{a_n\}$. However, $\alpha a + (1-\alpha)b < b_M$, which contradicts $b_M = \vee_{n \geq M} a_n$.

Case (1.d.ii): Given Reny (2011, Lemma A.18), it suffices to show that if f_n is a sequence of nondecreasing strategies converging in the L^1-norm to f, then $\vee_{n \geq m} f_n$ also converges to f as m goes to infinity. So let f_n be such sequence. From Reny (2011, Lemma A.12), it follows that f_n converges π_i-almost everywhere to f. Given that A_i is locally complete and using Reny (2011, Lemma A.18) again, it follows that $\vee_{n \geq m} f_n(t_i)$ converges to $f(t_i)$ for π_i-almost every t_i as m goes to infinity, which implies L^1-norm convergence. □

Given Lemmas B.1, B.2, and B.3, if the set of nondecreasing strategies M_i has monotonically contractible order intervals, then it is order-convex.

Lemma 6.3. *Under the assumptions of Theorem 6.1–(1), the set of nondecreasing strategies M_i for every player i is order-convex.*

Proof. From Reny (2011, Lemmas A.3 and A.15), it follows that if $[f_i, f_i']$ is an order interval in M_i, then $h\colon [0,1] \times [f_i, f_i'] \to [f_i, f_i']$ given by

$$h(\alpha, g_i) = \begin{cases} g_i(t_i) & \text{if } \Phi_i(t_i) \leq \alpha\,; \\ f_i'(t_i) & \text{otherwise}\,. \end{cases}$$

is a monotone contraction. Thus, M_i is order-convex. □

Notice that each player i's best reply is closed with respect to the supremum, by assumption, and closed with respect to the monotone contraction, by construction. Thus, given Lemmas 6.2 and 6.3, the existence of an equilibrium in nondecreasing pure strategies follows from Theorem 3.1.

Appendix A. Proofs for Section 2

A.1. Proof of Lemma 2.7.

Proof. If X is separable, metric space, then it can be isometrically embedded as a subset of a Banach space Y. It suffices to construct a retraction $r\colon X \to Y$.

For each $y \in Y \setminus X$, let $\varphi(y) = \inf\{\|y - x\|_Y \colon x \in X\}$. Define the correspondence $F\colon Y \setminus X \twoheadrightarrow X$ by

$$F(y) = \{x \in X \colon \|y - x\|_Y < 2\varphi(y)\}\,.$$

Because $\varphi(y) > 0$ for every $y \in Y \setminus X$, it follows that $F(y)$ is nonempty. Moreover, F has open lower sections. Thus, if X^* is a countable dense subset of X, then $\{F^{-1}(x) \colon x \in X^*\}$ is a countable open cover of $Y \setminus X$. Let $\mathcal{U}$ be a locally finite refinement, and let $\{\pi_U \colon U \in \mathcal{U}\}$ be a partition of unity subordinated to it. For each $U \in \mathcal{U}$, there is at least one $x \in X^*$ such that $U \subseteq F^{-1}(x)$; let x_U denote such x. For every $y \in Y \setminus X$, we identify the collection $\pi(y) = \{\pi_U(y) \colon U \ni y\}$ with the corresponding point in the simplex $|\{x_U \colon U \ni y\}|$. By Walker's carrier lemma, there exists a continuous function $f \colon |\Delta| \to |\Gamma|$, such that for every finite subset $Y' \subseteq Y$, $f(|Y'|) \subseteq |\mathcal{H}(Y')^\Gamma|$. Define the function $r \colon Y \setminus X \to X$ by

$$r(y) = h(f(\pi(y))\,.$$

Extend the function r to X by setting $r(x) = x$ for every $x \in X$.

Since $r|_{Y \setminus X}$ and $r|_X$ are continuous by construction, it suffices to check that, for every sequence $(y_n) \subseteq Y \setminus X$ converging to some $x \in X$, the sequence $(r(y_n))$ converges to $r(x) = x$. But, for every n, if $x' \in \operatorname{supp} \pi(y_n)$, then $d(y_n, x') < 2\varphi(y_n)$. As n goes to infinity, $\varphi(y_n)$ converges to zero. Because the hulling is small, that implies that $\mathcal{H}(\operatorname{supp} \pi(y_n))$ converges to x. Further, because the monotone realization is local, $h(f(\pi(y_n)))$ converges to x. □

Appendix B. Locally complete semilattices

We now investigate the relationship between these structures and locally complete metric semilattices.

Lemma B.1. *If X is a locally complete, metric semilattice and $\mathcal{H}$ is the family of all finite subsemilattices, then $\mathcal{H}$ is a small hulling.*

Proof. Let Y_n be a sequence of finite sets converging to $x \in X$, and let U be a neighborhood of x. Since X is locally complete, there is a neighborhood W of x such that every nonempty $Y \subseteq W$ has a least upper bound in U. Suppose that $Y_n \subseteq W$, as is the case for large n. Then $\{\, y_1 \vee \cdots \vee y_k \colon y_1, \ldots, y_k \in Y_n \,\}$ is a subsemilattice that is contained in any subsemilattice that contains Y_n, so it is $\mathcal{H}(Y_n)$, and each of its elements is contained in U. Thus $\mathcal{H}(Y_n) \subseteq U$ for large n. □

The notion of a order-convexity is a straightforward generalization of the path-connected metric-lattices extensively studied in Anderson (1959), McWaters (1969), Lawson (1969), Lawson and Williams (1970), and Gierz et al. (1980). It arises quite naturally. An *order interval* in X is a set defined by

$$[x, x'] = \{\, y \in X \colon x \leq y \leq x' \,\},$$

for some $x \leq x'$. We say that the order interval $[x, x']$ is *monotonically contractible* if there is a contraction $\ell\colon [0,1] \times [x, x'] \to [x, x']$ such that if $\alpha \leq \alpha'$, then $\ell(\alpha, y) \leq \ell(\alpha', y)$ for every $y \in [x, x']$. The next lemma shows that every metric semilattice with monotonically contractible order intervals has a monotone realization.

Lemma B.2. *If X is metric semilattice with monotonically contractible order intervals, then there exists a continuous function $h\colon |\Gamma| \to X$ such that $h(|Y|) \subseteq [\wedge Y, \vee Y]$ for every finite chain $Y \subseteq X$.*

Proof. Notice that for any finite set Y the hull $\mathcal{H}(Y)$ is the set $\{\vee Z \colon Z \subseteq S,\ Z \neq \emptyset\}$. We will use the following fact: any continuous function from the boundary of a cell to a contractible space can be continuously extended across the entire cell. We will construct the monotone realization h by induction on the skeletons of Γ. Recall that the n-skeleton $\Gamma^{(n)}$ is the subcomplex consisting of the simplexes of Γ of dimension n or less. For every vertex x in $\Gamma^{(0)}$, let $h(x) = x$. For each simplex Y in $\Gamma^{(1)}$, choose a monotone path $\ell\colon [0,1] \to [\wedge Y, \vee Y]$, and let $h(\pi) = \ell(\pi(\wedge Y))$ for every $\pi \in |Y|$. Notice that $h(\delta_{\wedge Y}) = \wedge Y$ and $h(\delta_{\vee Y}) = \vee Y$. Therefore, h is well-defined and continuous on $|\Gamma^{(1)}|$. Further, $h(|Y|) \subseteq [\wedge Y, \vee Y]$ for every 1-simplex Y in $\Gamma^{(1)}$. The inductive hypothesis is that $h\colon |\Gamma^{(n)}| \to X$ is continuous and $h(|Y|) \subseteq [\wedge Y, \vee Y]$ for every n-simplex Y in $\Gamma^{(n)}$. Now, suppose Z is an $(n+1)$-simplex. For every proper face Y of Z, $h(|Y|) \subseteq [\wedge Y, \vee Y] \subseteq [\wedge Z, \vee Z]$. Therefore, $h(|\operatorname{Bd} Z|) \subseteq [\wedge Z, \vee Z]$. Since Z is a cell and $[\wedge Z, \vee Z]$ is contractible, h can be continuously extended over $|Z|$ in such a way that $h(|Z|) \subseteq [\wedge Z, \vee Z]$. Since the map $h\colon |\Gamma| \to X$ is continuous if and only if it is continuous on each simplex, it follows that h is a monotone realization. □

If additionally X is locally complete, then the monotone realization constructed in the proof of Lemma B.2 is local.

Lemma B.3. *Suppose X is a locally complete, metric semilattice with a monotone realization $h\colon |\Gamma| \to X$. If $h(|Y|) \subseteq [\wedge Y, \vee Y]$ for every finite chain $Y \subseteq X$, then h is local.*

Proof. Let $Y_n \subseteq X$ be a sequence of nonempty finite chains converging to $x \in X$, and let $\underline{Y}_n = \wedge Y_n$ and $\overline{Y}_n = \vee Y_n$. For every n take any $x_n \in [\underline{Y}_n, \overline{Y}_n]$. Birkhoff's identity implies that for every n

$$\begin{aligned}\lfloor \underline{Y}_n - \overline{Y}_n \rfloor &= \lfloor \overline{Y}_n \vee x_n - \underline{Y}_n \vee x_n \rfloor + \lfloor \overline{Y}_n \wedge x_n - \underline{Y}_n \wedge x_n \rfloor \\ &= \lfloor \overline{Y}_n - x_n \rfloor + \lfloor x_n - \underline{Y}_n \rfloor ,\end{aligned}$$

where $\lfloor z \rfloor = z \vee (-z)$ denotes the absolute value of z. Since $\overline{Y}_n$ and $\underline{Y}_n$ both converge to x and X is locally complete, it follows that $\lfloor \overline{Y}_n - \underline{Y}_n \rfloor$ converges to 0. Therefore, $\lfloor \overline{Y}_n - x_n \rfloor$ and $\lfloor x_n - \underline{Y}_n \rfloor$ also converge to 0. Because $\overline{Y}_n$ converges to x, it follows that x_n converges to x too. □

Appendix C. Proofs for Sections ?? and 4

In all three auctions described in this section, the bidders' type and action spaces are subsets of Euclidean spaces. When required, we equip these spaces with the Lebesgue σ-algebra and the Lebesgue measure λ. In particular, this means that density functions on types are absolutely continuous with respect to the Lebesgue measure. Moreover, under these assumptions, the partial order on strategies induced either by the pointwise supremum or the pointwise infimum is measurable.

C.1. **All-pay auction.**

We first describe the bidder-specific set of strategies K_i. We then show that, using the sufficient conditions from Lemmas B.2 and B.1, it satisfies the requirements of Theorem 3.1.

Fix a bidder i. To describe the set K_i, let N_i^+ denote the set of indexes k such that the weighted valuation $v_i(t_i, t_{-i}) f(t_{-i} \mid t_i)$ is nondecreasing on the interval $\mathcal{I}_i^n$, that is, define

$$N_i^+ = \{n \colon v_i(t_i, t_{-i}) f(t_{-i} \mid t_i) \text{ is nondecreasing over } \mathcal{I}_i^n\} .$$

Notice that, given Assumption B.1, N_i^+ consists of a finite collection of indexes. Likewise, define N_i^- to be the set of indexes k such that the weighted valuation $v_i(t_i, t_{-i}) f(t_{-i} \mid t_i)$ is nonincreasing on the interval $\mathcal{I}_i^n$, that is,

$$N_i^- = \{n \colon v_i(t_i, t_{-i}) f(t_{-i} \mid t_i) \text{ is nonincreasing over } \mathcal{I}_i^n\} .$$

We may take N_i^+ and N_i^- to be disjoint. We define K_i to be the set of measurable functions from $T_i = [\underline{\tau}, \overline{\tau}]$ to $\mathcal{B}$ that are nondecreasing over $\mathcal{I}_i^n$ when $n \in N_i^+$ and nonincreasing over $\mathcal{I}_i^n$ when $n \in N_i^-$. Formally, define

$$\begin{aligned} K_i = \{f \colon\, & f|_{\mathcal{I}_i^n} \text{ is nondecreasing for every } n \in N_i^+ \\ & \text{and } f|_{\mathcal{I}_i^n} \text{ is nonincreasing for every } n \in N_i^-\} . \end{aligned} \tag{1}$$

As defined, K_i is a closed subset of functions of bounded variation, with a uniform total variation bound of $| \vee \mathcal{B} - \wedge \mathcal{B}| \times (|N_i^+| + |N_i^-|)$. Thus, by Helly's selection theorem, it is a L^1-norm compact subset of measurable functions. The following lemmas show that K_i satisfies the conditions required to apply Theorem 3.1.

Lemma C.1. *The subset of strategies K_i is a locally complete, metric semilattice.*

Proof. The set K_i, endowed with the L^1-norm, is clearly a metric semilattice. It only remains to show that it is locally complete. Given Reny (2011, Lemma A.18), it suffices to show that if g_k is a sequence of strategies in K_i converging in the L^1-norm to f, then $\vee_{k \geq m} g_k$ also converges to g as m goes to infinity. Let g_k be such sequence. Fix $n \in N_i^+$ and consider the function given by $g_k \mathbf{1}_{\mathcal{I}_i^n}$, where $\mathbf{1}_E$ is the indicator function of $E \subseteq T_i$. Because g_k is nondecreasing on $\mathcal{I}_i^n$, from Reny (2011, Lemma A.12), it follows that $g_k \mathbf{1}_{\mathcal{I}_i^n}$ converges almost everywhere to $g \mathbf{1}_{\mathcal{I}_i^n}$. Applying the same argument to $-g_k \mathbf{1}_{\mathcal{I}_i^n}$ for $n \in N_i^-$ yields that $g_k \mathbf{1}_{\mathcal{I}_i^n}$ converges almost everywhere to $g \mathbf{1}_{\mathcal{I}_i^n}$ for every $n \in N_i^+ \cup N_i^-$. Since there is a finite number of subintervals, it follows that $g_k = \sum_n g_k \mathbf{1}_{\mathcal{I}_i^n}$ converges almost everywhere to $g = \sum_n g \mathbf{1}_{\mathcal{I}_i^n}$. Given the real numbers are locally complete, applying Reny (2011, Lemma A.18) again, it follows that $\vee_{k \geq m} g_k(t_i)$ converges to $g(t_i)$ for almost every t_i as m goes to infinity. Therefore, $\vee_{k \geq m} g_k$ converges to g in the L^1-norm. □

In view of Lemma B.1, we record the following corollary of this result.

Corollary C.2. *The family $\mathcal{H}$ of all finite subsemilattices of K_i is a small hulling.*

The next lemma shows that the order intervals of K_i are monotonically contractible.

Lemma C.3. *The subset of strategies K_i has monotonically contractible order intervals.*

Proof. Let $[g_i', g_i'']$ be an order interval in K_i. Define the function $h\colon [0,1] \times [g_i', g_i''] \to [g_i', g_i'']$ by

$$h(\alpha, g_i) = \begin{cases} g_i''(t_i) & \text{if } t_i \in I_i^n \text{ with } k \in N_i^+ \text{ and } | \vee I_i^n - t_i| \leq \alpha | \vee I_i^n - \wedge I_i^n| \,, \\ g_i''(t_i) & \text{if } t_i \in I_i^n \text{ with } k \in N_i^- \text{ and } |t_i - \wedge I_i^n| \leq \alpha | \vee I_i^n - \wedge I_i^n| \,, \\ g_i(t_i) & \text{otherwise} \,. \end{cases}$$

The function h is the required monotone contraction. □

As a result of Lemmas B.2 and B.3, we have the following corollary of this result.

Corollary C.4. *The subset of strategies K_i is order-convex.*

The next two lemmas check that the best response correspondence also satisfies the conditions of the theorem.

Lemma C.5. *The intersection of the best response correspondence $B_i(\beta_{-i})$ with K_i is nonempty for every strategy profile of other bidders β_{-i}.*

Proof. Fix a profile of strategies for other players β_{-i}. We show that the interim best response correspondence

$$B_i(\beta_{-i} \mid t_i) = \arg\max_{b \in \mathcal{B}} V_i(b \mid t_i, \beta_{-i})$$

has a selection in K_i. Consider the selection $g_i(t_i) = \vee B_i(\beta_{-i} \mid t_i)$. It is well-defined because $\mathcal{B}$ is finite. Moreover, it is measurable because the pointwise partial order is measurable. The proof now procedes by contradiction to show that g_i is in K_i. Suppose $g_i \notin K_i$. Then there exist $t_i' > t_i$, both in some subinterval $\mathcal{I}_i^n$, such that either (i) $g_i(t_i) > g_i(t_i')$ and $n \in N_i^+$, or (ii) $g_i(t_i') > g_i(t_i)$ and $n \in N_i^-$.

Consider case (i). Because g_i is defined as the maximum interim best response, it follows that $g_i(t_i) \notin B_i(\beta_{-i} \mid t_i')$. Thus

$$V_i(g_i(t_i') \mid t_i', \beta_{-i}) - V_i(g_i(t_i) \mid t_i', \beta_{-i}) > 0\,. \tag{2}$$

Furthermore,

$$\begin{aligned}
&V_i(g_i(t_i) \mid t_i', \beta_{-i}) - V_i(g_i(t_i') \mid t_i', \beta_{-i}) \\
&= \int_{[\underline{\tau},\overline{\tau}]^{I-1}} [\alpha_i(g_i(t_i), \beta_{-i}(t_{-i})) - \alpha_i(g_i(t_i'), \beta_{-i}(t_{-i}))] v_i(t_i', t_{-i}) f(t_{-i} \mid t_i')\, dt_{-i} \\
&\qquad - g_i(t_i) + g_i(t_i')\,.
\end{aligned}$$

Since the allocation mapping α_i is positive and nondecreasing in its first argument and $v_i f_i$ is positive and nondecreasing in bidder i's signal, it follows that

$$\begin{aligned}
&\int_{[\underline{\tau},\overline{\tau}]^{I-1}} [\alpha_i(g_i(t_i), \beta_{-i}(t_{-i})) - \alpha_i(g_i(t_i'), \beta_{-i}(t_{-i}))] v_i(t_i', t_{-i}) f(t_{-i} \mid t_i')\, dt_{-i} \geq \\
&\qquad \int_{[\underline{\tau},\overline{\tau}]^{I-1}} [\alpha_i(g_i(t_i), \beta_{-i}(t_{-i})) - \alpha_i(g_i(t_i'), \beta_{-i}(t_{-i}))] v_i(t) f(t_{-i} \mid t_i)\, dt_{-i}\,,
\end{aligned}$$

and hence

$$V_i(g_i(t_i) \mid t_i', \beta_{-i}) - V_i(g_i(t_i') \mid t_i', \beta_{-i}) \geq V_i(g_i(t_i) \mid t_i, \beta_{-i}) - V_i(g_i(t_i') \mid t_i, \beta_{-i})\,,$$

However, optimality also implies that

$$V_i(g_i(t_i) \mid t_i, \beta_{-i}) - V_i(g_i(t_i') \mid t_i, \beta_{-i}) \geq 0\,,$$

and hence

$$V_i(g_i(t_i) \mid t_i', \beta_{-i}) - V_i(g_i(t_i') \mid t_i', \beta_{-i}) \geq 0\,,$$

which contradicts equation (2).

Consider now case (ii). Because g_i is defined as the maximum interim best response, it follows that

$$V_i(g_i(t_i) \mid t_i, \beta_{-i}) - V_i(g_i(t_i') \mid t_i, \beta_{-i}) > 0\,. \tag{3}$$

Furthermore,

$$\begin{aligned} &V_i(g_i(t_i') \mid t_i, \beta_{-i}) - V_i(g_i(t_i) \mid t_i, \beta_{-i}) \\ &= \int_{[\underline{\tau},\overline{\tau}]^{I-1}} [\alpha_i(g_i(t_i'), \beta_{-i}(t_{-i})) - \alpha_i(g_i(t_i), \beta_{-i}(t_{-i}))] v_i(t) f(t_{-i} \mid t_i)\, dt_{-i} \\ &\qquad - g_i(t_i') + g_i(t_i)\,. \end{aligned}$$

Since the allocation mapping α_i is positive and nondecreasing in its first argument and $v_i f_i$ is positive and nonincreasing in bidder i's signal, it follows that

$$\begin{aligned} &\int_{[\underline{\tau},\overline{\tau}]^{I-1}} [\alpha_i(g_i(t_i'), \beta_{-i}(t_{-i})) - \alpha_i(g_i(t_i), \beta_{-i}(t_{-i}))] v_i(t) f(t_{-i} \mid t_i)\, dt_{-i} \geq \\ &\qquad \int_{[\underline{\tau},\overline{\tau}]^{I-1}} [\alpha_i(g_i(t_i'), \beta_{-i}(t_{-i})) - \alpha_i(g_i(t_i), \beta_{-i}(t_{-i}))] v_i(t_i', t_{-i}) f(t_{-i} \mid t_i')\, dt_{-i}\,, \end{aligned}$$

and hence

$$V_i(g_i(t_i') \mid t_i, \beta_{-i}) - V_i(g_i(t_i) \mid t_i, \beta_{-i}) \geq V_i(g_i(t_i') \mid t_i', \beta_{-i}) - V_i(g_i(t_i) \mid t_i', \beta_{-i})\,,$$

However, optimality also implies that

$$V_i(g_i(t_i') \mid t_i', \beta_{-i}) - V_i(g_i(t_i) \mid t_i', \beta_{-i}) \geq 0\,.$$

and hence

$$V_i(g_i(t_i') \mid t_i, \beta_{-i}) - V_i(g_i(t_i) \mid t_i, \beta_{-i}) \geq 0\,,$$

which contradicts equation (3). □

Lemma C.6. *The intersection of the best response correspondence $B_i(\beta_{-i})$ with K_i is order-convex for every β_{-i}.*

Proof. Fix β_{-i}. Since the intersection of $B_i(\beta_{-i})$ with K_i is a closed subset of K_i and K_i is locally complete, it follows that $B_i(\beta_{-i}) \cap K_i$ is locally complete. Further, the best response correspondence B_i is closed with respect to the monotone contraction h constructed in Lemma C.3. Therefore, $B_i(\beta_{-i}) \cap K_i$ is order-convex for every β_{-i}. □

Corollaries C.2 and C.4, together with Lemmas C.5 and C.6 imply that the assumptions of Theorem 3.1 are satisfied for the all-pay auction when K_i is the set of strategies of bounded variation defined as by equation (1). Therefore, the all-pay auction has a Bayesian equilibrium in which bidders play strategies in K_i.

C.2. **First-price auction with interdependent values.** We first describe the bidder-specific set of strategies K_i, and show that it is order-convex. We then show that the best responses satisfy the requirements of Theorem 3.1.

Fix a bidder i. For every subset of indexes $L \subseteq \{1, \ldots, K\}$, define the following set of types of bidder i:

$$T_i^L = \left\{ t \in [\underline{\tau}, \overline{\tau}]^K : \frac{\partial v_i}{\partial t_{i\ell}}(t) \geq 0 \text{ if } \ell \in L \text{ and } \frac{\partial v_i}{\partial t_{i\ell}}(t) < 0 \text{ if } \ell \notin L \right\} .$$

Notice that each T_i^L is a (Borel) measurable subset of $[\underline{\tau}, \overline{\tau}]^K$. Furthermore, they constitute a partition of bidder i's type space, since $\cup_L T_i^L = [\underline{\tau}, \overline{\tau}]^K$ and $T_i^L \cap T_i^{L'} = \emptyset$ whenever $L \neq L'$. Thus each $t_i \in [\underline{\tau}, \overline{\tau}]^K$ is an element of T_i^L for one and only one $L \subseteq \{1, \ldots, K\}$.

Define K_i to be the set of (equivalence classes of) measurable functions from $[\underline{\tau}, \overline{\tau}]^K$ to $\mathcal{B}$ such that their restriction to each T_i^L is nondecreasing in $t_{i\ell}$ if $\ell \in L$ and nonincreasing in $t_{i\ell}$ if $\ell \notin L$. We consider K_i to be a subset of the set of real-valued, measurable functions over $[\underline{\tau}, \overline{\tau}]^K$ under the L^1-norm topology. We next show that the subset K_i is compact.

Lemma C.7. *The set K_i is L^1-norm compact.*

Proof. If $\frac{\partial v_i}{\partial t_{i\ell}}(t) = 0$ for every $t \in [\underline{\tau}, \overline{\tau}]^K$, then the result is straightforward. So we may assume that is not the case. Let $g_n \in K_i$ be a sequence of functions in K_i. By the diagonal argument, there exists a subsequence n_k such that $\lim_{n_k} g_{n_k}(r) = h(r)$ exists for every r in a countable dense subset of $[\underline{\tau}, \overline{\tau}]^K$. Define the function $g\colon [\underline{\tau}, \overline{\tau}]^K \to \mathcal{B}$ by

$$g(t) = \wedge \{ h(\tilde{t}) \colon \tilde{t}_\ell > t_\ell \text{ if } t \in T_i^L \text{ and } \ell \in L, \text{ and } \tilde{t}_\ell < t_\ell \text{ if } t \in T_i^L \text{ and } \ell \notin L \} .$$

By construction, $g \in K_i$. Moreover, $\lim_{n_k} g_{n_k}(t) = g(t)$ for continuity points of g. Theorem 7 of Brunk et al. (1956) and the fact that the set of roots of a nonzero polynomial function has zero Lebesgue measure imply that the set of of discontinuity points of g has zero Lebesgue measure. And since the distribution of bidder i's types is absolutely continuous with respect to the Lebesgue measure, it follows that g_{n_k} converges to g in the L^1-norm. □

Partially order K_i by the almost everywhere pointwise order, whereby

$$g_i \geq g_i' \qquad \Longleftrightarrow \qquad g_i(t_i) \geq g_i'(t_i) \qquad \lambda\text{-a.e}\,,$$

where λ denotes the Lebesgue measure. With this partial order, the set K_i is a locally complete semilattice.

Lemma C.8. *The set K_i with the almost everywhere pointwise order is a locally complete lattice.*

Proof. Given Reny (2011, Lemma A.18), it suffices to show that if g_n is a sequence of strategies in K_i converging in the L^1-norm to f, then $\vee_{n\geq m} g_n$ also converges to g as m goes to infinity. Let g_n be such sequence. Fix T_i^L and consider the function given by $g_n \mathbf{1}_{T_i^L}$, where $\mathbf{1}_E$ is the indicator function of $E \subseteq T_i$. Because g_n is nondecreasing in $t_{i\ell}$ for $\ell \in L$ and nonincreasing in $t_{i\ell}$ for $\ell \notin L$, from Reny (2011, Lemma A.12), it follows that $g_n \mathbf{1}_{T_i^L}$ converges almost everywhere to some $g\mathbf{1}_{T_i^L}$. Applying the same argument to each $L' \subseteq \{1, \ldots . K\}$ yields that $g_n \mathbf{1}_{T_i^{L'}}$ converges almost everywhere to $g\mathbf{1}_{T_i^{L'}}$ for every L'. Since there is a finite number of subsets of $\{1, \ldots, K\}$, it follows that $g_n = \sum_L g_n \mathbf{1}_{T_i^L}$ converges almost everywhere to $g = \sum_n g\mathbf{1}_{T_i^L}$. Given the real numbers are locally complete, applying Reny (2011, Lemma A.18) again, it follows that $\vee_{n\geq m} g_n(t_i)$ converges to $g(t_i)$ for almost every t_i as m goes to infinity. Therefore, $\vee_{n\geq m} g_n$ converges to g in the L^1-norm. □

Let $\mathcal{H}_i$ denote the collection of all finite subsemilattices of K_i. The next lemma shows that $\mathcal{H}_i$ is a small hulling.

Lemma C.9. *The collection $\mathcal{H}_i$ of all finite subsemilattices of K_i is a small hulling.*

Proof. It follows from Lemmas C.8 and B.1. □

Finally, we define a monotone realization for K_i. For the purposes of this example, a monotone realization is a continuous function $h\colon |\Gamma| \to K_i$ from order simplexes in Γ to K_i.

For every $L \subseteq \{1, \ldots, K\}$ and $c \in [0, 1]$, define the following measurable set of bidder i's types:

$$E(c, L) = \{t \in [\underline{\tau}, \overline{\tau}]^K \colon t_{i\ell} \leq (1-c)\underline{\tau} + c\overline{\tau} \text{ if } \ell \in L, \text{ and } t_{i\ell} \geq c\underline{\tau} + (1-c)\overline{\tau} \text{ if } \ell \notin L\}\,.$$

Notice that the collection $\{E(c,L)\colon c \in [0,1]\}$ is an increasing chain of measurable subsets of bidder i's type space that reflects the ordering induced by the partial derivatives of the valuation function in T_i^L. Further, the Lebesgue measure of each set $E(c,L)$ is $\lambda(E(c,L)) = c(\overline{\tau} - \underline{\tau})$, $E(0,L)$ is a singleton for every L, and $E(1,L) = [\underline{\tau},\overline{\tau}]^K$ for every L. Therefore, it follows that, for every $t_i \in [\underline{\tau},\overline{\tau}]^K$, there exists one $L \subseteq \{1,\dots,K\}$ such that $t_i \in E(1,L) \cap T_i^L = T_i^L$.

If $Y \in \Gamma$ is a simplex in the order complex of K_i, then Y consists of a finite chain in K_i. Thus the elements in Y can be identified with the ordered vector $Y = (g^1,\dots,g^n)$, with $g^1 \leq \cdots \leq g^n$. And a point x in the geometric realization $|Y|$ can be written as $x = (x_{g^1}, x_{g^2}, \dots, x_{g^n})$. We can now define the monotone realization $h\colon |\Gamma| \to K_i$ by

$$h(x)(t_i) = \begin{cases} g^1(t_i) & \text{if } t_i \in E(x_{g^1},L) \cap T_i^L\,, \\ g^2(t_i) & \text{if } t_i \in [E(x_{g^1}+x_{g^2},L) \setminus E(x_{g^1},L)] \cap T_i^L\,, \\ \cdots & \\ g^n(t_i) & \text{if } t_i \in [E(1,L) \setminus E(\sum_{k=1}^{n-1} x_{g^k},L)] \cap T_i^L\,. \end{cases}$$

That the function h is continuous follows from the Pasting Lemma and the fact that the distribution of bidders' types is absolutely continuous with respect to the Lebesgue measure. The next lemma establishes that h is a local monotone realization.

Lemma C.10. *The monotone realization h is local.*

Proof. It follows from Lemmas C.8 and B.3. □

All that is left to show is that the best response correspondence satisfies the conditions required by Theorem 3.1. We denote by $V_i(b \mid t_i, \beta_{-i})$ bidder i's interim payoff, given by

$$V_i(b \mid t_i, \beta_{-i}) = \int_{[\underline{\tau},\overline{\tau}]^{(N-1)K}} \rho_i(b, \beta_{-i}(t_{-i}))[v_i(t) - b] \prod_{j \neq i} f_j(t_j)\, \mathrm{d}\, t_{-i}\,.$$

Lemma C.11. *Fix a bid profile $\beta_{-i} \in K_{-i}$ of players other than i. If $B_i(\beta_{-i})$ is bidder i's best response to β_{-i}, then $B_i(\beta_{-i}) \cap K_i$ is nonempty and order-convex.*

Proof. Fix a profile β_{-i} of bids for players other than i. We first show that the intersection $B_i(\beta_{-i}) \cap K_i$ is not empty. Let B_i denote the interim best response correspondence, defined by

$$B_i(\beta_{-i} \mid t_i) = \arg\max_{b \in \mathcal{B}} V_i(b \mid t_i, \beta_{-i})\,,$$

and consider the selection $g_i(t_i) = \vee B_i(\beta_{-i} \mid t_i)$. It is well-defined because $\mathcal{B}$ is finite. Moreover, it is measurable because the pointwise partial order is measurable.

Suppose $t_i, t_i' \in T_i^L$ are such that $t_{i\ell} \geq t_{i\ell}'$ for $\ell \in L$ and $t_{i\ell} \leq t_{i\ell}'$ for $\ell \notin L$. It suffices to show that if $b \leq b'$ and $b \in B_i(\beta_{-i} \mid t_i)$ and $b' \in B_i(\beta_{-i} \mid t_i')$, then $b' \in B_i(\beta_{-i} \mid t_i)$.

$$
\begin{aligned}
&V_i(b' \mid t_i, \beta_{-i}) - V_i(b \mid t_i, \beta_{-i}) = \\
&\qquad \int [\rho_i(b', \beta_{-i}(t_{-i})) - \rho_i(b, \beta_{-i}(t_{-i}))] v_i(t_i, t_{-i}) \prod_{j \neq i} f_j(t_j) \, \mathrm{d}\, t_{-i} \\
&\qquad - \int [\rho_i(b', \beta_{-i}(t_{-i})) b' - \rho_i(b, \beta_{-i}(t_{-i}) b] \prod_{j \neq i} f_j(t_j) \, \mathrm{d}\, t_{-i} \, .
\end{aligned}
$$

Since $v_i(t_i, t_{-i}) \geq v_i(t_i, t_{-i})$ for every t_{-i} and $\rho_i(b', \beta_{-i}(t_{-i})) - \rho_i(b, \beta_{-i}(t_{-i})) \geq 0$, it follows that

$$
\begin{aligned}
&V_i(b' \mid t_i, \beta_{-i}) - V_i(b \mid t_i, \beta_{-i}) \geq \\
&\qquad \int [\rho_i(b', \beta_{-i}(t_{-i})) - \rho_i(b, \beta_{-i}(t_{-i}))] v_i(t_i', t_{-i}) \prod_{j \neq i} f_j(t_j) \, \mathrm{d}\, t_{-i} \\
&\qquad - \int [\rho_i(b', \beta_{-i}(t_{-i})) b' - \rho_i(b, \beta_{-i}(t_{-i}) b] \prod_{j \neq i} f_j(t_j) \, \mathrm{d}\, t_{-i} \, .
\end{aligned}
$$

Therefore,

$$
V_i(b' \mid t_i, \beta_{-i}) - V_i(b \mid t_i, \beta_{-i}) \geq V_i(b' \mid t_i', \beta_{-i}) - V_i(b \mid t_i', \beta_{-i}) \geq 0 \, .
$$

Because $b \in B_i(\beta_{-i} \mid t_i)$, it follows that $b' \in B_i(\beta_{-i} \mid t_i)$.

Since the intersection $B_i(\beta_{-i}) \cap K_i$ is a closed subset of K_i that is closed with respect to the hulling from Lemma C.9 and with respect to the monotone realization h from Lemma C.10, it follows that $B_i(\beta_{-i}) \cap K_i$ is order-convex. □

Therefore, by Theorem 3.1, this auction has an equilibrium in K.

Appendix D. Proofs for Section 5

D.1. **Proof of Theorem 5.1.** For convenience, we repeat the assumptions made in Section 5:

(1) Each player i's type space $T_i = [\underline{\tau}_i, \overline{\tau}_i]^{M_i} \times [\underline{\tau}_i, \overline{\tau}_i]^{M_i'}$ is a nondegenerate Euclidean cube, with the coordinate-wise partial order.
(2) Each player i's types are distributed according to the probability density f_i over T_i, not necessarily everywhere positive, but independently of other players' types.

(3) Each player i's set of actions $A_i = [\underline{\alpha}_i, \overline{\alpha}_i]^{N_i} \times [\underline{\alpha}_i, \overline{\alpha}_i]^{N_i'}$ is an Euclidean cube, endowed with the coordinate-wise partial order.
(4) Each player i's payoff function $u_i \colon A \times T \to \mathbb{R}$ is bounded, measurable in t, and continuous in a.

In addition, Theorem 5.1 makes the following assumptions on each player i's payoff function $u_i \colon A \times T \to \mathbb{R}$.

(a) Take any given $a_{-i} \in A_{-i}$ and $t \in T$. If $a, a' \in A_i$, then

$$u_i(a, a_{-i}; t) - u_i((a_{N_i} \wedge a'_{N_i}, a_{N_i'} \vee a'_{N_i'}), a_{-i}; t) \geq 0$$
$$\Rightarrow \qquad u_i((a_{N_i} \vee a'_{N_i}, a_{N_i'} \wedge a'_{N_i'}), a_{-i}; t) - u_i(a', a_{-i}; t) \geq 0\,.$$

(b) Take any given $a_{-i} \in A_{-i}$ and $t_{-i} \in T_{-i}$. Suppose $a_i, a_i' \in A_i$ are such that $a_{ik} \geq a'_{ik}$ for $k \in N_i$ and $a_{ik} \leq a'_{ik}$ for $k \in N_i'$; and $t_i, t_i' \in T_i$ are such that $t_{i\ell} \geq t'_{i\ell}$ for $\ell \in M_i$ and $t_{i\ell} \leq t'_{i\ell}$ for $\ell \in M_i'$, then

$$u_i(a_i, a_{-i}; t_i', t_{-i}) - u_i(a_i', a_{-i}; t_i', t_{-i}) \geq 0$$
$$\Rightarrow \qquad u_i(a_i, a_{-i}; t_i, t_{-i}) - u_i(a_i', a_{-i}; t_i, t_{-i}) \geq 0\,.$$

With these assumptions in mind, we can now prove Theorem 5.1. Take any player i. Define K_i to be the set of (equivalence classes of) measurable functions s_i from $T_i = [\underline{\tau}, \overline{\tau}]^{M_i} \times [\underline{\tau}, \overline{\tau}]^{M_i'}$ to $A_i = [\underline{\alpha}, \overline{\alpha}]^{N_i} \times [\underline{\alpha}, \overline{\alpha}]^{N_i'}$ that satisfy two conditions:

(1) If $k \in N_i$, then $t_i \mapsto s_{ik}(t_i)$ is nondecreasing in $t_{i\ell}$ whenever $\ell \in M_i$, and nonincreasing in $t_{i\ell}$ whenever $\ell \in M_i'$.
(2) If $k \in N_i'$, then $t_i \mapsto s_{ik}(t_i)$ is nonincreasing in $t_{i\ell}$ whenever $\ell \in M_i$, and nondecreasing in $t_{i\ell}$ whenever $\ell \in M_i'$.

Endow K_i with the L^1-norm topology. Partially order K_i by the almost everywhere pointwise order $\geq_i$ whereby $g_i \geq_i g_i'$ if and only if for μ_i-almost every t_i

$$g_{ik}(t_i) \geq g'_{ik}(t_i) \text{ if } k \in N_i\,, \text{ and}$$
$$g_{ik}(t_i) \leq g'_{ik}(t_i) \text{ if } k \in N_i'\,,$$

where $g_{ik}(t_i)$ denotes the projection of the vector $g_i(t_i)$ onto the k-th coordinate of the action space $A_i = [\underline{\alpha}, \overline{\alpha}]^{N_i} \times [\underline{\alpha}, \overline{\alpha}]^{N_i'}$.

Lemma D.1. *The set K_i is L^1-norm compact and locally complete.*

Proof. The set K_i is homomorphic and lattice isomorphic[15] to the set of monotone functions from T_i to A_i. By Lemma A.13 in Reny (2011, p. 540), K_i is compact. By Lemma 6.2, the set K_i is locally complete. □

Let $\mathcal{H}_i$ denote the collection of all finite subsemilattices of $(K_i, \geq_i)$. The next lemma shows that $\mathcal{H}_i$ is a small hulling.

Lemma D.2. *The collection $\mathcal{H}_i$ of all finite subsemilattices of $(K_i, \geq_i)$ is a small hulling.*

Proof. It follows from Lemmas D.1 and B.1. □

Finally, we define a monotone realization for K_i. Recall that a monotone realization is a continuous function $h\colon |\Gamma| \to K_i$ from order simplexes in Γ to K_i. Let $\mathbf{1}_M$ denote the indicator vector, in which the ℓ-th entry is 1 if $\ell \in M$ or 0 if $\ell \notin M$. Notice that $\mathbf{1}_M \cdot \mathbf{1}_M$ denotes the number of non-zero entries in M, with $M = M_i, M_i'$. For every $c \in [0,1]$, define the following measurable set of player i's types:

$$E(c) = \{t_i \in T_i\colon (\mathbf{1}_{M_i} - \mathbf{1}_{M_i'}) \cdot t_i \leq (1-c)(\underline{\tau} - \overline{\tau})(\mathbf{1}_{M_i} \cdot \mathbf{1}_{M_i}) + c(\overline{\tau} - \underline{\tau})(\mathbf{1}_{M_i'} \cdot \mathbf{1}_{M_i'})\}\,.$$

Notice that the collection $\{E(c)\colon c \in [0,1]\}$ is an increasing chain of measurable subsets of bidder i's type space that reflects the ordering induced by the natural order of $[\underline{\tau}, \overline{\tau}]^{M_i}$ and the dual order of $[\underline{\tau}, \overline{\tau}]^{M_i'}$. Further, notice that $E(0)$ is a singleton, and $E(1) = T_i$.

If $Y \in \Gamma$ is a simplex in the order complex of K_i, then Y consists of a finite chain in K_i. Thus the elements in Y can be identified with the ordered vector $Y = (g^1, \ldots, g^n)$, with $g^1 \leq_i \cdots \leq_i g^n$. And a point x in the geometric realization $|Y|$ can be written as $x = (x_{g^1}, x_{g^2}, \ldots, x_{g^n})$. We can now define the monotone realization $h\colon |\Gamma| \to K_i$ by

$$h(x)(t_i) = \begin{cases} g^1(t_i) & \text{if } t_i \in E(x_{g^1})\,, \\ g^2(t_i) & \text{if } t_i \in E(x_{g^1} + x_{g^2}) \setminus E(x_{g^1})\,, \\ \cdots & \\ g^n(t_i) & \text{if } t_i \in E(1) \setminus E(\sum_{k=1}^{n-1} x_{g^k})\,. \end{cases}$$

That the function h is continuous follows from the Pasting Lemma and the fact that the distribution of players' types is absolutely continuous with respect to the Lebesgue measure. The next lemma establishes that h is a local monotone realization.

[15]If $(X, \vee, \wedge)$ and $(X', \vee, \wedge)$ are lattices, then a lattice isomorphism is a bijective mapping $\kappa\colon X \to X'$ such that $\kappa(x \vee y) = \kappa(x) \vee \kappa(y)$ and $\kappa(x \wedge y) = \kappa(x) \wedge \kappa(y)$.

Lemma D.3. *The monotone realization h is local.*

Proof. The desired result follows from an application of Lemmas D.1 and B.3. □

Together, Lemmas D.2 and D.3 imply that K_i is order-convex, which is recorded in the following corollary.

Corollary D.4. *The set $(K_i, \geq_i)$ is an order-convex subset of strategies of player i.*

Assumptions (1)–(4) imply that Assumptions A.1 and A.2 are satisfied. All that is left to show is that the best response correspondence satisfies the conditions required by Theorem 3.1. We denote by $V_i(a \mid t_i, s_{-i})$ player i's interim payoff, given by

$$V_i(a \mid t_i, s_{-i}) = \int_{T_{-i}} u_i(a, s_{-i}(t_{-i}); t) \prod_{j \neq i} f_j(t_j) \, \mathrm{d}\, t_{-i} \,.$$

Lemma D.5. *Fix a profile s_{-i} of strategies of players other than i and a type $t_i \in T_i$ of bidder i. Let $B_i(s_{-i} \mid t_i)$ be bidder i's interim best response to s_{-i} when her type is t_i. If $a, b \in B_i(s_{-i} \mid t_i)$, then $(a_{N_i} \vee b_{N_i}, a_{N_i'} \wedge b_{N_i'}) \in B_i(s_{-i} \mid t_i)$.*

Proof. Suppose $a, b \in B_i(s_{-i} \mid t_i)$. By Assumption (a) of Theorem 5.1,

$$V_i(b \mid t_i, s_{-i}) - V_i(a_{N_i} \wedge b_{N_i}, a_{N_i'} \vee b_{N_i'} \mid t_i, s_{-i}) \geq 0$$

implies that

$$V_i(a_{N_i} \vee b_{N_i}, a_{N_i'} \wedge b_{N_i'} \mid t_i, s_{-i}) - V_i(a \mid t_i, s_{-i}) \geq 0\,.$$

Since $a, b \in B_i(s_{-i} \mid t_i)$, it follows that $(a_{N_i} \vee b_{N_i}, a_{N_i'} \wedge b_{N_i'}) \in B_i(s_{-i} \mid t_i)$, which completes the proof. □

Lemma D.6. *Fix a profile s_{-i} of strategies of players other than i and let $B_i(s_{-i})$ be player i's best response to s_{-i}. The intersection $B_i(s_{-i}) \cap K_i$ is nonempty.*

Proof. Consider the following selection of the interim best-response $B_i(s_{-i} \mid t_i)$ of player i given her type t_i and strategies of other players s_{-i}:

$$g_i(t_i) = \{(\vee a_{N_i}, \wedge a_{N_i'})\colon a \in B_i(s_{-i} \mid t_i)\}\,.$$

It is well-defined because A_i is a compact sublattice of $\mathbb{R}^{N_i \cup N_i'}$. Moreover, it is measurable because the pointwise partial order is measurable.

Suppose $t_i, t_i' \in T_i$ are such that $t_{i\ell} \geq t_{i\ell}'$ for every $\ell \in M_i$ and $t_{i\ell} \leq t_{i\ell}'$ for every $\ell \in M_i'$. It suffices to show that if $a \in B_i(s_{-i} \mid t_i)$ and $b \in B_i(s_{-i} \mid t_i')$, then $(a_{N_i} \vee b_{N_i}, a_{N_i'} \wedge b_{N_i'}) \in B_i(s_{-i} \mid t_i)$. Since $b \in B_i(s_{-i} \mid t_i')$,

$$V_i(b \mid t_i', s_{-i}) - V_i(a_{N_i} \wedge b_{N_i}, a_{N_i'} \vee b_{N_i'} \mid t_i', s_{-i}) \geq 0\,.$$

By Assumption (b) of Theorem 5.1,

$$V_i(a_{N_i} \vee b_{N_i}, a_{N_i'} \wedge b_{N_i'} \mid t_i, s_{-i}) - V_i(a \mid t_i, s_{-i}) \geq 0\,.$$

Since $a \in B_i(s_{-i} \mid t_i)$, it follows that $(a_{N_i} \vee b_{N_i}, a_{N_i'} \wedge b_{N_i'}) \in B_i(s_{-i} \mid t_i)$, which completes the proof. □

Lemma D.7. *Fix a profile s_{-i} of strategies of players other than i and let $B_i(s_{-i})$ be player i's best response to s_{-i}. The intersection $B_i(s_{-i}) \cap K_i$ is closed with respect to the hulling $\mathcal{H}_i$.*

Proof. The desired result follows from Lemmas D.5 and D.6. □

Lemma D.8. *Fix a profile s_{-i} of strategies of players other than i and let $B_i(s_{-i})$ be player i's best response to s_{-i}. The intersection $B_i(s_{-i}) \cap K_i$ is closed with respect to the monotone realization h.*

Proof. This follows from the construction of the monotone realization. □

Together, Lemmas D.6-D.8 imply the following corollary, which allows us to apply Theorem 3.1 to show that this Bayesian game has an equilibrium in K.

Corollary D.9. *Fix a profile $s_{-i} \in K_{-i}$ of strategies of players other than i and let $B_i(s_{-i})$ be player i's best response to s_{-i}. The intersection $B_i(s_{-i}) \cap K_i$ is nonempty and order-convex.*

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