A FIXED POINT THEOREM FOR CLOSED-GRAPHED DECOMPOSABLE-VALUED CORRESPONDENCES

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Abstract. Extending the fixed-point theorem of Cellina–Fryszkowski [1, 7], which is for functions on decomposable sets, to decomposable-set-valued correspondences has been an unresolved challenge since the early attempt of Cellina, Colombo, and Fonda [2]. Motivated by the fixed point problem of Reny [12] arising in Bayesian games, this paper proves such a theorem.

1. Introduction

Let (S, Σ, μ) be a measure space and let L be a Banach space. Denote by $L_1(\mu, L)$ the Banach space of all (equivalence classes of) Bochner μ-integrable functions $f\colon S \to L$. A subset $F \subseteq L_1(\mu, L)$ is *decomposable* if for every $f, g \in F$ and every $E \in \Sigma$, the function $h\colon S \to L$ defined by

$$h(s) = \begin{cases} f(s) & \text{if } s \in E\,, \\ g(s) & \text{if } s \notin E\,, \end{cases}$$

is also in F. The following fixed-point property of decomposable sets is known.

Theorem 1.1 (Cellina-Fryszkowski [1, 7]). *Suppose that (S, Σ, μ) is nonatomic. Let $F \subseteq L_1(\mu, L)$ be a nonempty closed decomposable set and let $\psi\colon F \to F$ be a continuous function. If there is a norm-compact set $X \subseteq F$ satisfying $\psi(F) \subseteq X$, then there exists $f \in X$ such that $f = \psi(f)$.*

The proof of this theorem is straightforward once it is established that the smallest closed decomposable set $D \subseteq F$ containing X is an absolute retract in the sense of [5]. Indeed, this means that there is a continuous function $r\colon L_1(\mu, L) \to D$ satisfying $r(f) = f$ for all $f \in D$. So the function $\psi \circ r$ maps the closed convex hull of X into X, which is norm compact, and thus has a fixed point in X that in turn is a fixed point of ψ.

Decomposable sets arise in a variety of applications. The canonical role of the fixed-point properties of decomposable sets in Bayesian game theory was highlighted in the recent work of Reny [12]. This role is simple to explain in the context of two person symmetric Bayesian games. In such games we have a nonatomic probability space (S, Σ, μ) and an *action space* A that is a norm compact subset of a Banach space L. There is a jointly measurable *utility function* $u\colon A \times A \times S \to \mathbb{R}$. The value $u(a, b, s)$ is the payoff that a player gets if she plays actions $a \in A$, the other player plays action $b \in A$, and state s realizes. The two players need to choose an

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integrable function $f\colon S \to A$ called a *strategy*. Let F be the set of all strategies, this set is closed and decomposable. A *symmetric Nash equilibrium* is a strategy $f \in F$ satisfying:

$$f \in \arg\max_{g\in F} \int_S u(g(s), f(s), s)\, d\mu(s)\,.$$

So letting $B\colon F \twoheadrightarrow F$ be the correspondence $f \mapsto \arg\max_{g\in F} \int_S u(g(s), f(s), s)\, d\mu(s)$ we notice that $B(f)$ is a decomposable set for every $f \in F$. Thus, the game has a symmetric Nash equilibrium if and only if B has a fixed point $f \in B(f)$. The problem with applying Theorem 1.1 is that in most of the interesting games the mapping B is not a function.

In this way we are concretely motivated to extend Theorem 1.1 to set-valued mappings. However, such an extension has been an unresolved challenge and as yet there is no known analogous theorem for the decomposable-set-valued maps. The question was initially addressed in [2] who proved a fixed-point theorem that turned out to be vacuous (see [3]). Indeed, the difficulty is that no non-singleton decomposable set is compact if the underlying measure space is nonatomic and the condition $\psi(F) \subseteq X$ in Theorem 1.1 cannot be satisfied if ψ is non-singleton but decomposable valued.

The present note proves a fixed point theorem for set-valued maps that are decomposable valued, which has significant applications in Bayesian games (see for instance [9]). A main contribution is our approach, which is quite different from those taken in the extant literature, in particular it is different from the selection-theoretic methods of [2]. Briefly speaking, we utilize the metrizable compact spaces studied in [10, 11], and, by means of the Hausdorff–Alexandroff Theorem, we pull back the fixed-point problem to a sub-lattice set of the Riesz space $L_1([0,1],\mathbb{R})$, which is the lattice of monotone functions from $[0,1]$ to the Cantor-ternary set. A straightforward application of the Eilenberg and Montgomery fixed point theorem [6] gives us the result. This final step is informed by the proof in Reny [12].

2. The fixed point theorem

Let (S, Σ, μ) be an atomless probability space and let T be a nonempty topological space. Let $\mathcal{L}(S,T)$ be the set of all functions, not necessarily measurable, from S to T. Importantly, elements of $\mathcal{L}(S,T)$ are functions and not μ-equivalence classes of functions. Endow $\mathcal{L}(S,T)$ with the topology of pointwise convergence, the product topology. A subset X of $\mathcal{L}(S,T)$ is metrizable if it is a metrizable topological space when endowed with the topology of pointwise convergence. Sets (of measurable functions) that are compact and metrizable in the topology of pointwise convergence are extensively studied in [10, 11].

For any $E \in \Sigma$ and any pair of functions $f, g \in \mathcal{L}(S,T)$ let $g_E f \in \mathcal{L}(S,T)$ be define as follows:

$$g_E f(s) = \begin{cases} g(s) & \text{if } s \in E\,, \\ f(s) & \text{if } s \in S \setminus E\,, \end{cases}$$

for all $s \in S$. A subset $F \subseteq \mathcal{L}(S,T)$ is *decomposable* if for each $E \in \Sigma$ and $f, g \in F$ we have $g_E f \in F$. It is *sequentially closed* if for any sequence $f_n \in F$ that converges pointwise to $f \in \mathcal{L}(S,T)$ we have $f \in F$. An analogous definition defines a sequentially closed subset of $\mathcal{L}(S,T) \times \mathcal{L}(S,T)$.

Theorem 2.1. *Suppose that (S, Σ, μ) is nonatomic and that $F \subseteq \mathcal{L}(S,T)$ is a sequentially closed decomposable subset of $\mathcal{L}(S,T)$. Let $B\colon F \twoheadrightarrow F$ be a correspondence satisfying the following conditions:*

1. *$B(f)$ is a decomposable set for all $f \in F$.*
2. *B has a sequentially closed graph in $F \times F$.*
3. *For all $f \in F$, if $\mu(E) = 0$ and $g \in B(f)$, then $h_E g \in B(f)$ and $g \in B(h_E f)$ for all $h \in F$.*

If there is a set $X \subseteq F$ that is compact and metrizable satisfying $X \cap B(f) \neq \emptyset$ for all $f \in F$, then B has a fixed point $f^* \in B(f^*)$.

3. Proof of Theorem 2.1

Let $C \subseteq [0,1]$ be the Cantor ternary set. Let $\mathcal{L}([0,1], C)$ be the set of all functions from $[0,1]$ to C. A function $\theta\colon \mathcal{L}([0,1],C) \to \mathcal{L}(S,T)$ is *sequentially pointwise continuous* if for any sequence $f_n \in \mathcal{L}([0,1],C)$ converging pointwise to $f \in \mathcal{L}([0,1],C)$, the sequence $\theta(f_n)$ converges pointwise to $\theta(f)$ in $\mathcal{L}(S,T)$. Extending the notion of decomposability to subsets of $\mathcal{L}([0,1],C)$, we say that $D \subseteq \mathcal{L}([0,1],C)$ is *decomposable* if for any pair of functions $f, g \in D$ and Borel set $E \subseteq [0,1]$ we have $g_E f \in D$.

Lemma 3.1. *There is a function $\theta\colon \mathcal{L}([0,1],C) \to \mathcal{L}(S,T)$ satisfying the following:*

1. *θ is sequentially pointwise continuous.*
2. *θ maps the constant functions in $\mathcal{L}([0,1],C)$ onto X.*
3. *If $D \subseteq \mathcal{L}(S,T)$ is decomposable, then $\theta^{-1}(D)$ is decomposable.*
4. *If $f, g \in \mathcal{L}([0,1],C)$ differ on exactly a countable set of points in $[0,1]$, then $\theta(f)$ and $\theta(g)$ differ on a μ-zero measure set.*

Proof. Order the members of Σ as follows $E \leq E'$ if either $E = E'$ or $E \subseteq E'$ and $\mu(E) < \mu(E')$. Consider a maximal chain $\{E_\lambda\}$ of this ordering containing S and the empty set. Because μ is atomless $E_\lambda \mapsto \mu(E_\lambda)$ is a one to one onto mapping from $\{E_\lambda\}$ to $[0,1]$. So we can reindex the maximal chain by means of the identity $\lambda = \mu(E_\lambda)$. Let $\mathbb{Q}$ be the set of rational numbers in $[0,1]$ and for each $s \in S$ let

$$r(s) = \inf\{\lambda \in \mathbb{Q}\colon s \in E_\lambda\} = \sup\{\lambda \in \mathbb{Q}\colon s \notin E_\lambda\}\,.$$

This is a measurable function satisfying $\mu(r^{-1}(E)) = 0$ for any zero measure Borel subset of $[0,1]$.

Because X is compact and metrizable, the Hausdorff-Alexandroff Theorem says that there is a continuous function ψ mapping C onto X.

Define the $\theta\colon \mathcal{L}([0,1],C) \to \mathcal{L}(S,T)$ as follows:

$$\theta(f)(s) = \psi(f(r(s)))(s)\,,$$

for all $f \in \mathcal{L}([0,1],C)$ and $s \in S$. Note that

$$\theta(g_E f) = \theta(g)_{r^{-1}(E)}\theta(f)$$

for any $f, g \in \mathcal{L}([0,1],C)$ and $E \subseteq [0,1]$.

We prove that θ has the required properties:

1. If $f_n \in \mathcal{L}([0,1],C)$ is a sequence that converges pointwise to $f \in \mathcal{L}([0,1],C)$, then for any $\alpha \in [0,1]$ the sequence $\psi(f_n(\alpha))$ converges to $f(\alpha)$ in X. Thus, for all $s \in S$ the sequence $\psi(f_n(\alpha))(s)$ converges in T. This tells us that $\theta(f_n)$ converges pointwise to $\theta(f)$ in $\mathcal{L}(S,T)$.

(2) If $f(\alpha) = c$ for all $\alpha \in [0,1]$, then $\theta(f)(s) = \psi(c)(s)$ for all s.
(3) Let $D \subseteq \mathcal{L}(S,T)$ be a decomposable set. If $f, g \in \theta^{-1}(D)$ and E is a Borel subset of $[0,1]$, then $r^{-1}(E)$ is in Σ and $\theta(g_E f) = \theta(g)_{r^{-1}(E)}\theta(f) \in D$. Thus, $g_E f \in \theta^{-1}(D)$.
(4) If $f, g \in \mathcal{L}([0,1], C)$ and $g = h_E f$ for some zero measure Borel set $E \subseteq [0,1]$, then $\theta(g) = \theta(h_E f) = \theta(h)_{r^{-1}(E)}\theta(f)$ and $r^{-1}(E) = 0$.

□

Fix a function $\theta\colon \mathcal{L}([0,1], C) \to \mathcal{L}(S,T)$ satisfying the properties in Lemma 3.1. Let $\mathcal{M}$ be the set of monotone functions from $[0,1]$ to C. This is a sequentially compact set in the topology of pointwise convergence. Let $Y = \theta(\mathcal{M})$, which is also a sequentially compact subset of $\mathcal{L}(S,T)$, because θ is sequentially continuous. The set Y contains X because $\mathcal{M}$ contains the constant functions, and θ maps the constant functions onto X. We want to show that Y has the μ-fixed point property.

Fix a set valued mapping $B\colon F \twoheadrightarrow F$ that is decomposable, μ-sequentially upper hemicontinuous, and that satisfies the following:

(a) $Y \cap F \neq \emptyset$.
(b) $Y \cap B(f) \neq \emptyset$ for all $f \in Y \cap F$.

We need to show that B has a fixed point in Y.

Let $\mathcal{F} = \theta^{-1}(F)$, which is a subset of $\mathcal{L}([0,1], C)$, and note that it is decomposable and sequentially closed, because of properties (3) and (1), respectively, of Lemma 3.1. For each $f \in \mathcal{F}$ let

$$P(f) = \theta^{-1}(B(\theta(f)))\,.$$

We record the following properties of the mapping $P\colon \mathcal{F} \twoheadrightarrow \mathcal{F}$.

Lemma 3.2. *The following hold true:*

(1) *$\mathcal{F}$ is sequentially closed, decomposable, and $\mathcal{M} \cap \mathcal{F}$ is non-empty.*
(2) *For each $f \in \mathcal{F}$, the set $P(f)$ is sequentially closed and decomposable.*
(3) *P has a sequentially closed graph in $\mathcal{F} \times \mathcal{F}$.*
(4) *If $E \subseteq [0,1]$ is countable, then $g \in P(f)$ implies that $h_E g \in P(f)$ and $g \in P(h_E f)$ for all $h \in \mathcal{F}$.*
(5) *For any $f \in \mathcal{M} \cap \mathcal{F}$, the set $P(f) \cap \mathcal{M}$ is nonempty.*
(6) *If f is a fixed point of P, then $\theta(f)$ is a fixed point of B.*

Proof. (1) and (2) are consequences of (1) and (3) of Lemma 3.1. (3) is a consequence of (1) of Lemma 3.1. (4) follows from (4) of Lemma 3.1. (5) holds because $B(f) \cap Y$ is not empty for any $f \in Y \cap F$. Finally, (6) holds because if $f \in P(f)$, then $\theta(f) \in B(\theta(f))$. □

So our task now is to show that P has a fixed point in $\mathcal{M}$.

Let $\mathcal{Z} = \mathcal{F} \cap \mathcal{M}$, which is sequentially closed and nonempty by (1) of Lemma 3.2. Define the mapping $Q\colon \mathcal{Z} \to \mathcal{Z}$ by letting

$$Q(f) = P(f) \cap \mathcal{M}\,,$$

for all $f \in \mathcal{Z}$. This is a nonempty valued correspondence with sequentially closed graph in $\mathcal{Z} \times \mathcal{Z}$, because of (2) of Lemma 3.2. For any $f \in \mathcal{M}$ let $\overleftarrow{f}$ be the right continuous version of f; setting $\overleftarrow{f}(1) = 1$ for all $f \in \mathcal{M}$. Recall that $\overleftarrow{f}$ differs from f over a countable subset of $[0,1]$. Also, if $\overleftarrow{g} = \overleftarrow{f}$, then g differs from f on a countable subset of $[0,1]$. In particular, $Q(f) = Q(g)$ and if $f \in Q(h)$, then

$g \in Q(h)$. This is, as a result of property (1) of the definition of μ-sequentially graphed mappings and (4) of Lemma 3.1.

For $\mathcal{G} \subseteq \mathcal{M}$, we write $\mathcal{G}^{\leftarrow}$ for the set $\{f^{\leftarrow} : f \in \mathcal{G}\}$. For each $f \in \mathcal{Z}^{\leftarrow}$ choose an arbitrary $g \in \mathcal{Z}$ satisfying $g^{\leftarrow} = f$ and let

$$\tilde{Q}(f) = Q(f)^{\leftarrow} .$$

The mapping $\tilde{Q} \colon \mathcal{Z}^{\leftarrow} \twoheadrightarrow \mathcal{Z}^{\leftarrow}$ is nonempty valued, because Q is nonempty valued. Further, if f is a fixed point of $\tilde{Q}$, then any $g \in \mathcal{Z}$ satisfying $g^{\leftarrow} = f$ is a fixed point of Q, and the required fixed point of P. So we are done if we show that $\tilde{Q}$ has a fixed point.

Endow $\mathcal{M}$ with the pseudometric

$$\delta(f, g) = \int_0^1 |f(a) - g(a)| \, da .$$

Notice that $(\mathcal{M}^{\leftarrow}, \delta)$ and $(\mathcal{Z}^{\leftarrow}, \delta)$ are a compact metric spaces. Furthermore, $\tilde{Q}$ has a δ-closed graph in $\mathcal{Z}^{\leftarrow} \times \mathcal{Z}^{\leftarrow}$.

Order the set $\mathcal{M}^{\leftarrow}$ of right-continuous monotone functions by means of the pointwise ordering whereby $f \geq g$ if $f(\alpha) \geq g(\alpha)$ for all $\alpha \in [0, 1]$. The set $(\mathcal{M}^{\leftarrow}, \delta)$ is a δ-compact topological meet semilattice using the terminology in [8].

Let Γ be the set of all nonempty closed subsets of $(\mathcal{M}^{\leftarrow}, \delta)$ endowed with the metric induced by Hausdorff distances. For any $U \subseteq \mathcal{M}^{\leftarrow}$ we write $\inf U$ for the pointwise inf of the set of functions in U. This is a monotone right continuous function in $\mathcal{M}^{\leftarrow}$ and the infimum of the set U in the lattice $\mathcal{M}^{\leftarrow}$. Notice that $\inf U = \inf \overline{U}$, where $\overline{U}$ is the closure of U in $(\mathcal{M}^{\leftarrow}, \delta)$. This is because if f_n is a sequence in U that δ-converges to f, then it pointwise converges to some $g \in \mathcal{M}$ satisfying $g^{\leftarrow} = f$. But $g(a) \leq f(a)$ for all $a \in [0, 1]$. We will now show that the function $U \mapsto \inf U$ from Γ to $(\mathcal{M}^{\leftarrow}, \delta)$ is continuous.

Lemma 3.3. *If a sequence $U_n \in \Gamma$ converges to $U \in \Gamma$, then $\inf U_n$ converges to $\inf U$ in $(\mathcal{M}^{\leftarrow}, \delta)$.*

Proof. Let $f = \inf U$. For each n let $f_n = \inf U_n$. All of these are in $\mathcal{M}^{\leftarrow}$. Let f^* be an accumulation point in $(\mathcal{M}^{\leftarrow}, \delta)$ of f_n, by moving to a subsequence we shall suppose that f_n converges to $f^* \in \mathcal{M}^{\leftarrow}$. We want to show that $f^* = f$.

For each n let $V_n = \cup_{m \geq n} U_m$. Let $g_n = \inf V_n$ for each n, and recall that $g_n \in \mathcal{M}^{\leftarrow}$. The sequence g_n is increasing pointwise, so let g be $\sup\{g_n\}$ (taking the pointwise supermum), which is in $\mathcal{M}$ but not necessarily right continuous. Now U is in the closure of V_n for each n. Thus, $g_n = \inf(V_n \cup U) \leq \inf U = f$ for all n. In particular, $g(a) \leq f(a)$ for all $a \in [0, 1]$.

Let $\tilde{f}$ be the left continuous version of f, setting $\tilde{f}(0) = 0$. Suppose by way of contradiction that for some $a \in [0, 1]$ we have $g(a) < d_2 < d_1 < \tilde{f}(a)$. There is $\gamma > 0$ such that $\tilde{f}(a - \gamma) > d_1$. Pick n large enough such that $\delta(h, U) < (d_1 - d_2)\gamma$ for all $h \in V_n$. Pick $h \in V_n$ satisfying $h(a) < d_2$ and $h' \in U$ satisfying $\delta(h, h') < (d_1 - d_2)\gamma$. But $d_1 < \tilde{f}(b) \leq f(b) \leq h'(b)$ for all $a - \gamma \leq b$. Thus, $\delta(h, h') \geq (d_1 - d_2)\gamma$. This is impossible. We conclude that $g(a) \geq \tilde{f}(a)$ for all a. Thus, $g^{\leftarrow} = f$ and g_n converges to f.

Now note that $f_n \geq g_n$ for all n. For each a, for every $\epsilon > 0$, and n there is $m \geq n$ and $h \in U_m$ such that $|h(a) - g_n(a)| < \epsilon$. But $h(a) \geq f_m(a) \geq g_n(a)$. Thus, $f^* = f$. □

The result of Wojdysławski [13] (cf. [4]) tells us that when endowed with the metric induced by Hausdorff distances, the family of all nonempty closed subsets of a Peano continuum is an absolute retract. We employ this and the previous lemma to establish the next result.

Lemma 3.4. *If $\mathcal{G} \subseteq \mathcal{L}([0,1], C)$ is decomposable and $\mathcal{G} \cap \mathcal{M}$ is nonempty and sequentially closed, then $(\overleftarrow{(\mathcal{G} \cap \mathcal{M})}, \delta)$ is a compact absolute retract.*

Proof. The set $\overleftarrow{(\mathcal{G} \cap \mathcal{M})}$ is nonempty and compact. If $f, g \in \mathcal{G} \cap \mathcal{M}$, then $f \wedge g$ is monotone and differs from f, g on Borel sets. Thus, $f \wedge g$ is in $\mathcal{G} \cap \mathcal{M}$. Noting that $(f \wedge g)^{\leftarrow} = f^{\leftarrow} \wedge g^{\leftarrow}$, we see that $((\mathcal{G} \cap \mathcal{M})^{\leftarrow}, \delta)$ is a sub-semilattice of $\overleftarrow{\mathcal{M}}$. We show that it is locally connected, and thus a Peano continuum.

First, notice that if $f, g \in \mathcal{G} \cap \mathcal{M}$ and $f \geq g$, then $g_{[0,\alpha)}f \in \mathcal{G} \cap \mathcal{M}$ for all $\alpha \in [0,1]$. Thus, if $f, g \in (\mathcal{G} \cap \mathcal{M})^{\leftarrow}$, then $g_{[0,\alpha)}f \in (\mathcal{G} \cap \mathcal{M})^{\leftarrow}$ for all $\alpha \in [0,1]$.

If U_n is a neighborhood base in $((\mathcal{G} \cap \mathcal{M})^{\leftarrow}, \delta)$ of $f \in (\mathcal{G} \cap \mathcal{M})^{\leftarrow}$, then $\inf U_n$ converges to f by Lemma 3.3. Thus, $V_n = \{[\inf\{U_n\}, h] \colon h \in U_n\}$, where $[g, h] = \{h' \in \overleftarrow{(\mathcal{G} \cap \mathcal{M})} : g \leq h' \leq h\}$, is a neighborhood base at f. Let $g = \inf U_n$ and $h \in V_n$. For any $\alpha \in [0,1]$ the function $g_{[0,\alpha)}h$ is in V_n. Thus, V_n is path connected and $\overleftarrow{(\mathcal{G} \cap \mathcal{M})}$ is Peano continuum.

The collection Γ^* of nonempty closed subsets of $\overleftarrow{(\mathcal{G} \cap \mathcal{M})}$ is an absolute retract and the mapping $U \mapsto \inf U$ from Γ^* to $\overleftarrow{(\mathcal{G} \cap \mathcal{M})}$ is a continuous retract. This concludes the proof. $\square$

The metric space $(\overleftarrow{\mathcal{Z}}, \delta)$ is a compact absolute retract, and for each $f \in \overleftarrow{Z}$ the set $\tilde{Q}(f)$ is an absolute retract. Noting that $\tilde{Q}$ has a closed graph in $(\mathcal{Z}^{\leftarrow}, \delta) \times (\overleftarrow{\mathcal{Z}}, \delta)$, by the Eilenberg and Montgomery fixed point theorem [6] the correspondence $\tilde{Q}$ has a fixed point. This concludes the proof of Theorem 2.1.

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